Lucknow University Studies

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No. I.

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THE THEORY AND CONSTRUCTION OF NON-DIFFERENTIABLE FUNCTIONS.

PREFACE

A systematic account of the theory and construction of non-differentiable functions is not found in any published text book on the theory of functions. Hobson, in his *Theory of Functions of a Real Variable*, has given an account of Knopp's method of construction of non-differentiable functions by means of infinite series, but does not mention geometrical and arithmetical methods of obtaining such functions. Other text books contain isolated examples and do not attempt to give any general theory.

In this summary of a course of four lectures, delivered at the Lucknow University, my aim has been to include as comprehensive an account of developments relating to non-differentiable functions as is possible within a limited scope of about a hundred pages. Necessary proofs of the theorems have been given in some cases. In other cases the reader will have to consult the original sources cited.

The first lecture contains the history of attempts made by nineteenth century mathematicians to construct non-differentiable functions, as well as a brief account of the general method evolved by Dini of obtaining such functions in the form of infinite series. A proof of the non-differentiability of Weierstrass's function based on a method of M. B. Porter has been given. This proof is simpler and yet more powerful than the one ordinarily found in text books.

In the second lecture is given an account of several non-differentiable functions defined geometrically. Although some of these functions are multiple-valued yet they have been included for the sake of their historical importance, especially as they were originally given as examples of continuous nowhere differentiable functions.

The third lecture deals with the history of arithmetically-defined non-differentiable functions. A detailed discussion of the derivates of an example constructed by me in 1924 is included. Two other simple examples have also been discussed. In one of these the decimal representation of fractions is used for obtaining the definition of the function.

The fourth lecture is devoted to the discussion of the properties of non-differentiable functions, especially with regard to their oscillating nature and the existence of cusps and maxima and minima. Some theorems relating to the derivates of continuous functions, which have a direct bearing on our subject, have also been included.

A bibliography of original sources cited in the text is appended at the end. In the body of the book they are referred to by their serial numbers.

It is a pleasure to record here my indebtedness to Dr. Birbal Sahni for asking me to deliver these lectures and for making their publication possible. My thanks are also due to my friends and colleagues, Mr. M. L. Bhatia for preparing the diagrams and Mr. R. D. Misra for going through the proofs.

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FIRST LECTURE

FUNCTIONS DEFINED BY SERIES

1. **Early Notions.** In to-day's lecture I propose to deal with the historical aspect of our subject. Non-differentiable functions have played a great part in the refinement of our geometrical intuition, and were in part, if not wholly, responsible for the critical study of the notion of "limit" made by the nineteenth century mathematicians—a study which resulted in placing mathematical analysis on a sure and sound foundation. Up to the middle of the nineteenth century the notion of "function" was connected with the geometrical notion of "curve" defined as the path traced out by a moving point. This notion of curve implies that—

(*i*) The curve is continuous, because the moving point must pass through every point between any two points P and Q on the path;

(*ii*) the curve has a determinate tangent at each point, because a moving point has at every point of its path a determinate direction of motion;

(*iii*) the arc of the curve between any two points has a finite length, because the arc is described in finite time; and (*iv*) the curve does not make an indefinitely large number of oscillations in the neighbourhood of any point.

Examples of functions which could be expressed by a simple analytical formula, but which did not satisfy one or more of the above conditions, were known to mathematicians in the early nineteenth century, so that a definition of function, free from appeal to geometrical intuition, was a desideratum. Such a definition was provided by Dirichlet (1805—1859). Analytical definitions of continuity were given by Weierstrass (1815—1896), Cauchy (1784—1857) and Hankel. The notion of "magnitude" was given an analytical garb by Cantor (1845—1918) and Dedekind (1831—1916), who, independently of each other, developed their theories of the irrational number. By the help of their theories it is now possible to interpret all mathematical processes in terms of arithmetic.

2. Early History of the Calculus. Newton and Leibnitz are said to have discovered the *Differential Calculus*, but their ideas about it were incorrect and hazy. It is said that Newton did not believe in the results that he obtained by the help of the calculus until he had proved the same results by other methods. These mathematicians seem to have stumbled upon a very powerful tool, the exact nature of which they did not know. While Newton was suspicious, Leibnitz and his followers made free and sometimes indiscriminate use of the calculus in all branches of mathematics. So far as the use of the differential calculus, either as "the rate of flow" or as "the infinitesimal increment" is concerned, Newton and Leibnitz may be said to have made more extensive use of it than their predecessors, but, certainly, they were not the first to do so. It is well known that Fermat (1608—1665) had actually obtained the equation of the tangent in the form

$$Y-y = \underset{\substack{y' \to y \\ x' \to x}}{Lim} \frac{y-y'}{x-x'} (X-x).$$

3. **The Calculus in India.** The Hindu mathematicians appear to have been familiar with the idea of the "infinitesimal increment" from very early times. Mañjula (932 A.D.) has given the formula,

$$d\ (\ Sin\ \theta\ ) = Cos\ \theta\ d\ \theta$$

for the calculation of $Sin\ (\theta+\delta\theta)$, when $Sin\ \theta$ is known and $\delta\theta$ is small. Bhâskarâcârya (1150 A. D.) uses the term *tâtkâlikagati* to denote the infinitesimal increment, and has applied the conception to the evaluation of the instantaneous velocity of the moon and planets, and to a problem of maxima and minima, stating that the infinitesimal increment at a maximum is zero. Nîlakantha (c. 1500 A. D.) has shown that the increment of the increment of *Sin* θ varies as $-$ *Sin* θ, which stated in the notation of the calculus gives the formula

$$\frac{d^2}{d\theta^2}\ (Sin\ \theta) = -\ Sin\ \theta.$$

More extensive use of the method of the calculus seems to have been made by Hindu mathematicians during the fifteenth and sixteenth centuries. Talakulattura Nambutiri (1432 A. D.) has given the expansion of *Tan* x in ascending powers of x, which is usually attributed to Gregory (1671 A. D.). In another work, the *Sadratnamâlâ*, we find the well-known expansions in infinite series of *Sin* x and *Cos* x in powers of x.

I am sure that, if political conditions in India had been favourable, the method of the *Infinitesimal Calculus* would have been independently developed along indigenous lines in India.

4. **Continuity and Differentiability.** That continuity is necessary for the existence of a finite differential coefficient was probably known to Newton and Leibnitz, but whether it is or is not sufficient for differentiability seems to have been one of the outstanding questions till 1860, when it was finally answered in the negative by Weierstrass.

An attempt to prove that continuity was a sufficient condition for differentiability was made by Ampère (1) in 1806. Although Ampère's proof was defective yet his result was believed in by most mathematicians for a long time. Mention may be made of Duhamel, Bertrand and Gilbert ([30]) among those who definitely expressed their belief in Ampère's result. Even in the writings of such eminent mathematicians as Gauss, Cauchy, and Dirichlet there is nothing to show that they held a different opinion. Although they did not endorse Ampère's statement none of them seems to have had the conviction that a function which was everywhere continuous but nowhere differentiable could exist. Darboux in his memoir on "Discontinuous functions," published in 1875, mentions only one auditor, M. Bienaimé, who said that he was unconvinced by Ampère's proof.

5. **Riemann's Non-differentiable Function.** It was asserted by some pupils of Riemann that, in his lectures in the year 1861, he gave the function represented by the infinite series

$$\sum_{n=1}^{\infty} \frac{Sin\ (n^2 \pi x)}{n^2}$$

as an example of a continuous non-differentiable function. No proof of Riemann's assertion was ever published by him or any of his pupils,* while Paul du Bois-Reymond,

* In a letter to Du Bois-Reymond, dated 23rd November, 1873, Weierstrass says that Riemann was reported to have said that the proof would come from elliptic functions; see *Acta Math.*, Vol. 39, p. 199.

in a paper in *Crelle's Journal*, 1874, states without proof that Riemann's Function, for certain values of x, in any interval, ever-so-small, has no finite differential coefficient. The only writer who has considered the non-differentiability of Riemann's Function is G. H. Hardy ([32]). He has shown that Riemann's Function "is certainly not differentiable for any irrational and some rational values of x". Definite information is not available as regards the existence or non-existence of the differential coefficient at the other points. It can, however, be easily proved that Riemann's Function has an infinite differential coefficient with positive sign at the point $x = o$. Thus the function is not *totally* non-differentiable.

6. **Condensation of Singularities.** Methods of constructing functions which do not possess a differential coefficient at an everywhere dense set of points were given by Cantor ([15]) and Hankel ([34]). These methods, however, fail to give a non-differentiable function in the strict sense of the term.

7. **Weierstrass's Discovery.** The question whether a continuous no-where differentiable function could exist was finally solved by Weierstrass's discovery of the classical example

$$f(x) = \sum_{n=1}^{\infty} a^n \operatorname{Cos} (b^n \pi x),$$

where b is an odd integer, $o < a < 1$, and $ab > 1 + \frac{3\pi}{2}$.

Although discovered much earlier, and communicated by Weierstrass in his lectures, the function was first published in 1874 by Paul du Bois-Reymond ([24]). Due Bois-Reymond was awe-struck by Weierstrass's discovery and terms it as 'equally too strange for immediate perception as well as for critical understanding.'

8. **Attempts of Other Writers.** Darboux ([20]) in 1875 gave an example of a function which does not possess a finite differential coefficient. He makes no mention of Weierstrass's Function published in 1874, and was doubtless not aware of it.

Of earlier attempts to get a non-differentiable function may be mentioned those of Cellerier and Bolzano. There is reason to believe that the function

$$\sum_{n=1}^{\infty} \frac{Sin\ (a^n \pi x)}{a^n}$$

was discovered by Cellerier before 1850, as has been pointed out by G. C. Young ([100]). The function is, however, not non-differentiable in the strict sense of the term, as it possesses infinite differential coefficients at an everywhere dense set of points.*

9. **Bolzano's Function.** Bolzano's non-differentiable function was brought to light in 1921, when

*Hobson (36), pp. 406 – 7; also B. N. Prasad (64). G. C. Young (100) and A. Falanga (28) thought that Cellerier's function was completely non-differentiable.

its discovery, in a manuscript of Bolzano said to date from the year 1830, was first announced by M. Jasek in the sitting of the 16th December, 1921, of the Bohemian Society of Sciences. Proofs of the non-differentiability of the function have been supplied by K. Rychlik ([70]), G. Kowalewsky ([49, 50]), and A. N. Singh ([80]). From Jasek's paper it appears that Bolzano contented himself with establishing the want of a differential coefficient at an everywhere dense but enumerable set of points. Not only was Bolzano unaware of the complete non-differentiability of his function, but that he, at one time, held the wrong opinion that 'a continuous function must be differentiable for every value of the variable with the exception of *isolated values*' is evident from a footnote to Art. 37 of his book, "*Paradoxien des Unendlichen*," published in 1847-48.

10. The Effect of Weierstrass's Discovery. The publication of Weierstrass's example created a sensation in mathematical circles. The discovery was hailed by men of keen acumen like Du Bois-Reymond, but there were others who could not easily bring themselves round to believe in Weierstrass's demonstration. Evidence of this tendency is to be found in a comprehensive paper published by Wiener ([97]) in which he made a detailed study of Weierstrass's Function, and sought to prove that it possessed a differential coefficient at an everywhere dense set of points.

Besides supplying the answer to a question which had long been agitating the minds of mathematicians Weierstrass's discovery opened up a new field of research—the subject of non-differentiability—a subject which has exercised great charm on the minds of mathematicians. Indeed, there are few amongst mathematicians of note who have not contributed something to the subject.

11. **Work Relating to Weierstrass's Function.** A large number of papers on the subject of non-differentiability group around Weierstrass's Function, or the generalised series

$$\sum a_n \, Cos \, (b_n \pi x) \text{ and } \sum a_n \, Sin \, (b_n \pi x),$$

where the a's and b's are positive the series $\sum a_n$ is convergent, and the b's increase steadily with more than a certain degree of rapidity.

The conditions under which the series

$$\sum a^n \, Cos \, (b^n \pi x)$$

does not possess a differential coefficient, finite or infinite, were given by Weierstrass as:

$$o < a < 1, \; ab > 1 + \frac{3\pi}{2},$$

where b is an odd integer.

The only direct improvement known on this is Bromwich's [14]

$$o < a < 1, \; ab > 1 + \frac{3\pi}{2}(1 - a),$$

where b is odd.

For the non-existence of a finite differential coefficient several conditions have been given. Dini gives the condition (23):

$$ab \geqq 1,\ ab^2 > 1 + 3\pi^2,$$

Lerch (53):

$$ab > 1 \quad , ab^2 > 1 + \pi^2$$

and Bromwich (14):

$$ab \geqq 1,\ ab^2 > 1 + \frac{3\pi^2}{4}\ (1-a).$$

All these conditions presuppose that b is an odd integer. But Dini has shown that if

$$ab > 1 + \frac{3\pi}{2} \cdot \frac{1-a}{1-3a} \quad , \quad a < \tfrac{1}{3};$$

or

$$ab \geqq 1,\ ab^2 > 1 + 15\pi^2 \frac{1-a}{5-21a},\ a < \frac{5}{21},$$

this restriction may be removed.

The best result in this connection is due to G. H. Hardy (32, 33) who has shown that neither of the functions

$$\Sigma\ a^n\ Cos\ (b^n \pi x) \text{ or } \Sigma\ a^n\ Sin\ (b^n \pi x)$$

where $o < a < 1,\ b > 1$,

possesses a finite differential coefficient at any point in any case in which $ab \geqq 1$. It has been further shown by him that the result is untrue if the word 'finite' is omitted. It has also been shown that these functions possess cusps at everywhere dense sets of points.

12. The Series Definition. Attempts have been made by various writers to generalize Weierstrass's result by considering the function

$$f(x) = \sum_{1}^{\infty} U_n(x) \qquad (i)$$

instead of Weierstrass's function

$$W(x) = \sum_{1}^{\infty} a^n \, Cos \, (b^n \pi x). \qquad (ii)$$

Mention may be made of Faber who replaces $Cos\ (b^n \pi x)$ in (ii) by the function $\psi(b_n\, x)$, where $\psi(x)$ is a polygonal function of period 1, such that in (0, 1),

$$\psi(x) = x, \text{ for } o \leq x \leq \tfrac{1}{2};$$

$$\text{and } \psi(x) = 1 - x, \text{ for } \tfrac{1}{2} \leq x \leq 1.$$

The function actually considered by Faber (25, 26) is

$$\sum_{1}^{\infty} \frac{1}{10^n} \psi(2^{n!}\, x).$$

The general case $\sum a^n \psi(b^n\, x)$ has been shown by Knopp to be non-differentiable when $ab > 4$.

Van der Waerden (96) has recently given a simple definition of the above case when $a = 1/10$ and $b = 10$.

Let $f_n(x)$ denote the distance between x and the nearest number of the form $m/10^n$, where m is an integer. Then, it is easy to see that

$$\sum_1^{\infty} \frac{1}{10^n} \psi(10^n x) = \sum_1^{\infty} f_n(x).$$

The above series does not give a non-differentiable function in the strict sense of the term for it can be shown to possess infinite differential coefficients at an enumerable everywhere dense set of points.*

Non-differentiable functions defined by series as in (i) above have been studied by Dini and Knopp. An account of Knopp's method of construction is given in Hobson's *Theory of Functions of a Real Variable* (Cambridge, 1927), Vol. II. I shall give a summary of Dini's method.†

13. **Dini's Method.** A general method of construction of non-differentiable functions was given by Dini ([22]) in 1877. He considers the general series

$$f(x) = \sum_1^{\infty} U_n(x)$$

* See footnote (2) to Van der Waerden's paper. Titchmarsh (92) mentions this example as a non-differentiable function although he proves only the non-existence of a finite differential coefficient.

† Dini - Lüroth (28) and Hobson's *Theory of Functions*, first edition, Cambridge, 1907.

where $U_n(x)$ is continuous in (a, b) for all values of n, and the series $\Sigma U_n(x)$ converges everywhere in (a, b) and defines a continuous function. It is further assumed that, for each value of n, $U_n(x)$ possesses maxima and minima, such that the interval between each maximum and the next minimum is a number δ_n which diminishes indefinitely as n is indefinitely increased; and also that $U_n(x) = -U_n(x+\delta_n)$, so that all the maxima of $U_n(x)$ are equal to one another, the maxima and minima being equal in absolute value and opposite in sign. It is also assumed that, for finite n, $U_n(x)$ possesses finite differential coefficients of the first and second orders $U'_n(x)$ and $U''_n(x)$ everywhere in (a, b); and that the upper limits of $|U'_n(x)|$ and $|U''_n(x)|$ have finite values $\overline{U}_n$ and $\overline{\overline{U}}_n$.

Let D_m denote the excess of a maximum over a minimun of $U_m(x)$. Let a neighbourhood $(x, x+\epsilon)$ or $(x-\epsilon, x)$ on either side of x be chosen, m may be chosen so great that several oscillations of $U_m(x)$ are completed in the chosen neighbourhood. Let the point $x+h$ be taken at a maximum or minimum of $U_m(x)$ in $(x, x+\epsilon)$ or in $(x-\epsilon, x)$; and let it be the first maximum or minimum of $U_m(x)$ on the right or on the left of x, of which the distance from x is $\geq \frac{1}{2}\delta_m$. The condition

$$| U_m(x+h) - U_m(x) | \geq \tfrac{1}{2} D_m$$

is satisfied. We note that $| h | \leq \frac{3}{2}\delta_m$.

Writing $U_m(x+h)-U_m(x)=\frac{1}{2}\propto_m\nu_m D_m$, where ν_m is positive and >1, and $\propto_m=\pm 1$, its sign depending on x and m, and possibly on h, the incrementary ratio

$$\text{(i)}\qquad \frac{f(x+h)-f(x)}{h}=\frac{\nu_m\propto_m D_m}{2h}\left[1+\frac{2\eta'_m h}{D_m}\sum_{n=1}^{m-1}\overline{U_n}\right.$$
$$\left.+\frac{2\propto_m}{\nu_m}\,\frac{R_m(x+h)-R_m(x)}{D_m}\right],$$

where η'_m lies between 1 and -1, and $R_m(x)$ denotes the remainder after m terms of the series $\sum\limits_{1}^{\infty}U_n(x)$. Again let $x+h_1$ be the next following extreme point of $U_m(x)$ after $x+h$, so that h and h_1 have the same sign and $|h_1|>|h|$. The difference $U_m(x+h_1)-U_m(x)$, when it is not zero, has the sign opposite to that of

$$U_m(x+h)-U_m(x),$$

and therefore $U_m(x+h_1)-U_m(x)=-\frac{1}{2}\,\xi'_m\propto_m\nu_m D_m$, where $o\leq\xi'_m<1$.

Then we have

$$\text{(ii)}\qquad \frac{f(x+h)-f(x)}{h}-\frac{f(x+h_1)-f(x)}{h_1}$$

$$=\frac{\propto_m\nu_m D_m}{2h}\left[1+\xi'_m\frac{h}{h_1}+\frac{4h\,\eta''_m}{D_m}\sum_{n=1}^{m-1}\overline{U}_n+\frac{2\propto_m}{\nu_m}\right.$$

$$\left.\frac{R_m(x+h)-R_m(x)}{D_m}-\frac{2\propto_m h}{\nu_m h_1}\,\frac{R_m(x+h_1)-R_m(x)}{D_m}\right]$$

and also

$$=\frac{\alpha_m \nu_m D_m}{2h}\Big[1+\varepsilon'_m \frac{h}{h_1}+\zeta''_m \frac{h(h+h_1)}{D_m}\sum_{n=1}^{m-1}\overline{\overline{U}}_n+2\alpha_m\theta_m$$

$$\frac{R_m(x+h)-R_m(x)}{D_m}-2\alpha_m\theta'_m\frac{h}{h_1}\,\frac{R_m(x+h_1)-R_m(x)}{D_m}\Big]$$

where η''_m and ζ''_m lie between 1 and -1, and θ_m, θ'_m between 0 and 1.

Dini applies (*i*) and (*ii*) to get the following four forms of sufficient conditions for the non-differentiability of $f(x)=\sum_1^\infty U_n(x)$:

(A) If

(1) $\dfrac{\delta_m}{D_m}$ has the limit zero when m is indefinitely increased;

(2) $R_m(x+h)-R_m(x)$ has, for values of m greater than an arbitrarily chosen integer m', the same sign as α_m;

(3) $\dfrac{3\,\delta_m}{D_m}\sum_{n=1}^{m-1}\overline{U}_n$ remains numerically less than unity by more than some fixed difference;

then, $f(x)$ has at no point a differential coefficient, either finite or infinite.

(B) If

(1) $\dfrac{\delta_m}{D_m}$ has the limit zero when m is indefinitely increased;

(2) $|R_m(x+h)-R_m(x)|$ has a finite upper limit $2R'_m$ for all values of x, and $R_m(x+h)-R_m(x)$ has, for values of m greater than an arbitrarily chosen integer m', the same sign as α_m;

(3) $\dfrac{3\delta_m}{D_m}\sum\limits_{n=1}^{m-1}\overline{U}_n+\dfrac{4R'_m}{D_m}$ remains less than unity by more than some fixed difference;

then, $f(x)$ has at no point a differential coefficient, either finite or infinite.

(C) If

(1) $\dfrac{\delta_m}{D_m}$ has not the limit zero but remains less than some finite number, for all values of m;

(2) $Rm(x+h)-R_m(x)$ has the same sign as α_m and $R_m(x+h_1)-R_m(x)$ has the opposite sign;

(3) $\dfrac{6\delta_m}{D_m}\sum\limits_{n=1}^{m-1}\overline{U}_n$ or $6\dfrac{\delta^2{}_m}{D_m}\sum\limits_{n=1}^{m-1}\overline{\overline{U}}_n$

remains less than unity by more than some fixed difference;

then, $f(x)$ has nowhere a finite differential coefficient, although it may have an infinite one at some points.

(D) If

(1) $\frac{\delta_m}{D_m}$ has not the limit zero, but remains less than some finite number, for all values of m;

(2) $|R_m(x+h)-R_m(x)|$, $|R_m(x+h_1)-R_m(x)|$ never exceed a finite number $2\,R'_m$;

(3) $$\frac{6\ \delta_m}{D_m}\sum_{n=1}^{m-1}\overline{U}_n+\frac{32}{5}\frac{R'_m}{D_m} \text{ or } \frac{6\ \delta^2_m}{D_m}\sum_{n=1}^{m-1}\overline{\overline{U}}_n+\frac{32\ R'_m}{5\ D_m}$$

remains less than unity by more than a fixed difference; then $f(x)$ has nowhere a finite differential coefficient, although it may have an infinite one at some points.

It is easy to see that Weierstrass's Function is a particular case of the class of functions considered by Dini.

14. Proof of Weierstrass's result. I shall give a proof of the non-differentiability of Weierstrass's function, and also show that *almost everywhere* the function does not possess a progressive or a regressive derivative.

$$\text{Let } W(x) = \sum_{0}^{\infty} a^n \, Cos\,(b^n\pi x), \text{ where } |a| < 1,$$

and b is integral.

Setting $\delta x = 2k/b^{m+1}$, k integral, we get by applying the mean value theorem to the first m terms and a trigonometric identity to the $(m+1)th$ term,

$$(14.1)\quad \frac{\Delta W(x)}{\delta x} = -\pi \sum_{0}^{m-1} (ab)^n \, Sin \, b^n\pi(x+\epsilon\delta x)$$

$$-\pi \; (ab)^m \; \frac{Sin \; k\pi/b}{k\pi/b} \; Sin \; (b^m x + \tfrac{k}{b})\pi,$$

other terms vanishing on account of having a *sine* factor whose argument is a multiple of π.

Evidently the absolute value of the first m terms is less than

$$\pi \sum_{0}^{m-1} | \, ab \, |^n < \frac{\pi \, | \, ab \, |^m}{| \, ab \, | - 1}, \text{ if } | \, ab \, | > 1.$$

Now, if we suppose that $k \leq \frac{3}{4}b$, the last term in (14.1) is in absolute value

$$(14.2)\quad \geq \pi \, | \, ab \, |^m \frac{1}{\frac{3}{4}\pi\sqrt{2}} \left| \; Sin \; (b^m x + \tfrac{k}{b})\pi \right|$$

Let x be expressed in the scale of b as

$$x = \frac{c_1}{b} + \frac{c_2}{b^2} + \cdots\cdots + \frac{c_n}{b^n} + \cdots\cdots\cdots\cdots,$$

where the c's are integers such that $0 \leq c_n \leq b-1$.

$$\text{Then} \left| Sin \; (b^m x + \tfrac{k}{b})\pi \right| = \left| Sin \; (I + \tfrac{k}{b} + \frac{c_{m+1}}{b} + \ldots) \right|$$

$$= \left| \; Sin \; (\tfrac{k}{b} + \frac{c_{m+1}}{b} + \cdots\cdots \right|$$

because I is an integer.

It is now easy te see that two values of k, k_1 and k_2, not necessarily equal, can be found such that

$$\text{(A)} \quad 1 \geq Sin\ (b^m x + \frac{k_1}{b})\,\pi \geq \frac{1}{\sqrt{2}}$$

$$\text{(B)} \quad -\frac{1}{\sqrt{2}} \geq Sin\ (b^m x + \frac{k_2}{b})\,\pi \geq -1,$$

where k_1 and k_2 have *in general* opposite signs.

Thus for these values of k,

$$\left| \pi\,(ab)^m \frac{1}{\frac{3}{4}\pi\sqrt{2}}\ Sin\ (b^m x + \frac{k}{b})\,\pi \right| \geq \frac{\pi\,|\,ab\,|^m}{\frac{3}{2}\pi}.$$

The last term of (14.1) will then dominate in sign and magnitude the first m terms if

$$|\,ab\,| > 1 + \frac{3\pi}{2}.$$

Hence the right and left incrementary ratios which we are considering will become infinite with m but will always have opposite signs.

This proves that $W(x)$ has at no point a differential coefficient.

15. Non-existence of the derivative.

We have $b^m x\pi = (I + \frac{c_{m+1}}{b} + \frac{c_{m+2}}{b^2} + \ldots\ldots)\pi$.

That is,

$$Sin\ (b^m x + \frac{k}{b})\,\pi = \pm\ Sin\ \left(\frac{k}{b} + \frac{c_{m+1}}{b} + \cdots\right)\pi$$

(15.1) Now, if $\frac{1}{2} < \frac{c_{m+1}}{b} + \frac{c_{m+2}}{b^2} + \ldots\ldots < \frac{3}{4}$, it is easy to see that a positive k_1 can be chosen so that (A) holds and another positive k_2 can be chosen for which (B) holds. Thus at all those points x, whose representation is such that the condition (15.1) is realized for infinitely many values of m, the right hand incrementary ratio oscillates between $+\infty$ and $-\infty$.

(15.2) Similary, if the representation of x' be such that the condition :

$$\tfrac{1}{4} \leq \frac{c_{m+1}}{b} + \frac{c_{m+2}}{b^2} + \cdots\cdots < \tfrac{1}{2}$$

is realized for infinitely many values of m, it can be shown that at each point x' the left hand incrementary ratio oscillates between $+\infty$ and $-\infty$.

Further, it can be easily shown that the set of points $C[x]$ complementary to the set $[x]$ for each point of which (15.1) holds, is a *null set*, if $b \geq 8$.* The same remark holds for the set $C[x']$ complementary to the set $[x']$.

Thus we obtain the following result :

Except at a set of measure zero both the upper derivates of $W(x)$ are $+\infty$ and both the lower derivates $-\infty$.†

* If $b = 8$, the set $C[x]$ consists of points in whose representation 4 and 5 do not occur an infinite number of times. This set is made up of an enumerable number of perfect null sets, and is everywhere dense on the line.

†This result was proved by G. C. Young (100) by means of an elaborate and lengthy analysis.

16. Some Important Non-Differentiable Functions Defined by Series.

I now give a list of some of the important non-differentiable functions which have been studied.

(1) $\sum a^n \operatorname{Cos}(b^n \pi x)$ where $o < a < 1$, b an odd integer, $ab > 1 + \frac{3\pi}{2}$; or $ab > 1 + \frac{3\pi}{2}(1-a)$, is non-differentiable.

(Weierstrass, Dini, Lerch, Bromwich)

(2) $\sum n^{-2} \operatorname{Sin} n^2 x$ does not possess a finite differential coefficient.

(Riemann, Hardy)

(3) $\sum a^n \operatorname{Sin}(b^n \pi x)$, where $o < a < 1$, $b = 4m$, $ab > 1 + \frac{3\pi}{2}$, is non-differentiable whatever be the signs of the individual terms.

(Dini, Porter)

(4) $\sum a^n \operatorname{Sin}(b^n \pi x)$, where $o < a < 1$, $ab > 9$, is non-differentiable whether b be odd or even.

(Porter)

(5) $\sum a^n \operatorname{Sin}(b^n \pi x)$, where $o < a < 1$, $b = 4m+1$, $ab > 1 + \frac{3\pi}{2}$, is non-differentiable.

(Dini, Knopp)

(6) $\sum (-1)^n a^n \operatorname{Sin}(b^n \pi x)$, where $o < a < 1$, $b = 4m+3$, $ab > 1 + \frac{3\pi}{2}$, is non-differentiable.

(Knopp)

(7) $\Sigma \frac{a^n}{n!} \begin{matrix} Sin \\ Cos \end{matrix} (n!\pi x)$, $|a| > 1 + \frac{3\pi}{2}$,

is non-differentiable.

(Porter)

(8) $\Sigma \frac{a^n}{1.\ 3.\ 5 \dots (2n-1)}\ Cos\,(1.\ 3.\ 5 \dots (2n-1)\pi x)$,

where $|a| > 1 + \frac{3\pi}{2}$ is non-differentiable.

(Darboux, Dini)

(9) $\Sigma \frac{a^n}{1.\ 5.\ 9 \dots (4n+1)}\ Sin\ (1.\ 5.\ 9 \dots (4n+1)\pi x)$,

$a > 1 + \frac{3\pi}{2}$, is non-differentiable.

(Darboux, Dini)

(10) If $\Sigma \frac{a^r}{10^r}$ denote any non-terminating decimal,

$\Sigma \frac{a^r}{10^r} \begin{matrix} Sin \\ Cos \end{matrix} (10^{3r}\pi x)$ is non-differentiable.

(Porter)

(11) $\Sigma\, a^n \begin{matrix} Sin \\ Cos \end{matrix} (a^n \pi x)$ does not possess a finite differential coefficient.

(Cellerier, Hardy)

(12) If the periodic function $\psi(x) = x$, for $o \leq x \leq \frac{1}{2}$; $\psi(x) = 1 - x$ for $\frac{1}{2} \leq x \leq 1$, then $\Sigma\, a^n\, \psi(b^n x)$, where $o < a < 1$ and $ab > 4$ is non-differentiable,

(Faber, Knopp)

(13) If the periodic function $\varphi(x)=x$, for $o \leq x \leq \frac{1}{2}$, $\varphi(x)=1-x$ for $\frac{1}{2} \leq x \leq \frac{3}{2}$, $\varphi(x)=x-2$, for $\frac{3}{2} \leq x \leq 2$, then $\sum a^n \psi(b^n x)$, where $o < a < 1$, $b=4m+1$, $ab > 4$, is non-differentiable.

(Knopp)

(14) $\sum (-1)^n a^n \varphi(b^n x)$, where $o < a < 1$, $b=4m+3$, $ab > 4$, is non-differentiable.

(Knopp)

(15) $\sum a^n \mid Sin\ (b^n \pi x) \mid$, where $o < a < 1$, and $ab > 1 + \dfrac{3\pi}{2}$ is non-differentiable.

(Knopp)

(16) $x \sum a^n\ Sin\ (b^n \pi x)$, where $\mid a \mid < 1$, $\mid ab \mid > 1 + \dfrac{3\pi}{2}$ has a differential coefficient for $x=0$ but for no other value of x,

(Porter)

(17) $\sum \dfrac{x^n}{n!} \dfrac{Sin}{Cos} (n! \pi x)$ has differential coefficients between -1 and 1, and no differential coefficients if $\mid x \mid > 1 + \dfrac{3\pi}{2}$.

(Lerch, Porter)

SECOND LECTURE

FUNCTIONS DEFINED GEOMETRICALLY

1. It is well known that a continuous curve $\Phi(x)$ can be defined by a convergent sequence of polygonal curves $[\Phi_n(x)]$. This process has been used by various writers to construct non-differentiable functions. To illustrate the method I shall give the construction of some typical curves obtained by the above process, pointing out the main properties of each curve.

2. **Bolzano's Curve.** The first non-differentiable function defined geometrically is Bolzano's curve. The construction, which depends upon the successive stretching and deformation of straight lines, may be given* as follows:

Divide a straight line PQ (which we denote by F_0) into the two halves PM and MQ, and each of these again into four equal parts PP_1, P_1P_2, P_2P_3, P_3M, and MQ_1, Q_1Q_2, Q_2Q_3, Q_3Q. Now let Q_3 be carried to Q'_3, and P_3 to P'_3, as shown in Fig. 1, so that the join of $PP'_3MQ'_3Q$ gives a zig-zag consisting of four stretches. We note that the slope of each stretch is double that of the original stretch PQ. We denote

* The construction given here is a modification due to Kowalewsky (50); see Singh (80).

$PP'_3MQ'_3Q$ by F_1. Bolzano's fundamental operation consists in the stretching of the line PQ into the zig-zag form $PP'_3MQ'_3Q$ which is made up of four doubly steep stretches. By applying the fundamental operation

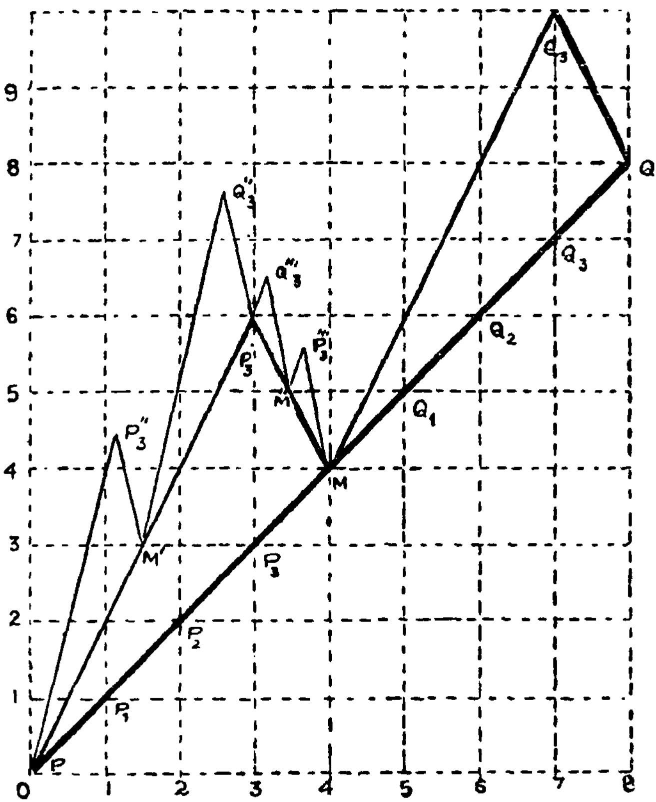


Fig. 1.

to each of the four parts of the zig-zag $PP'_3MQ'_3Q$ (curve F_1) Bolzano obtains a zig-zag line (curve F_2) consisting of sixteen parts (as shown in Fig. 1) each of which is again subjected to the fundamental operation, and so on.

The curves $F_0, F_1, F_2 \ldots$ so obtained converge towards a curve F, which referred to a horizontal x-axis and a vertical y-axis represents a single-valued, continuous, but nowhere differentiable function.

3. Proofs of the non-differentiability of Bolzano's function have been supplied by K. Rychlik (70) who shows that the function possesses cusps at an everywhere dense set of points and by G. Kowalewsky (49, 50). A. N. Singh (80) has given an analytical definition of Bolzano's function, as well as an analytical proof of its non-differentiability. For obtaining the analytical definition the numbers x in (0, 1) are represented as

$$(3.1) \quad x = \frac{3^{a_1}k_1}{8} + \frac{3^{a_2}k_2}{8^2} + \cdots + \frac{3^{a_n}k_n}{8^n} + \cdots\cdots,$$

where the a's and k's are defined as follows:

Let Φ_1 denote one of the numbers 3, 7; Φ_2 one of the numbers 4, 0; ψ_1 one of the numbers 1, 5; and ψ_2 one of the numbers 4, 0; then

$$a_1 = o, \text{ and } k_1 \text{ is } \Phi_1 \textit{ or } \Phi_2 \quad ,$$

and, in general,

if k_v is Φ_1, then $a_{v+1} = a_v$, and k_{v+1} is ψ_1 or ψ_2,
if k_v is Φ_2, then $a_{v+1} = a_v + 1$ and k_{v+1} is Φ_1 or Φ_2,
if k_v is ψ_1, then $a_{v+1} = a_v + 1$ and k_{v+1} is ψ_1 or ψ_2
if k_v is ψ_2, then $a_{v+1} = a_v$ and k_{v+1} is Φ_1 or Φ_2,

Considering the ending representations of x, we see that, when x consists of only one term, four points are obtained ; these being $\frac{3}{8}$, $\frac{7}{8}$, or $\frac{4}{8}$, $\frac{0}{8}$ according as k_1 is Φ_1 or Φ_2. When x consists of two terms, sixteen points are obtained by giving all possible values to k_1 and k_2. Of these sixteen points four (corresponding to $k_2=0$) have already been obtained. Of the remaining twelve, three lie in each of the four intervals into which $(0, 1)$ is divided by the points x consisting of one term only. Similarly it is easy to see that the aggregate of all the points consisting of n terms contains all the points whose representations run up to $(n-1)$ terms together with three points in each of the sub-intervals formed by these points. Thus it is obvious that each ending representation uniquely defines a point in $(0, 1)$, and that the aggregate of ending representations gives an enumerable everywhere dense set of points in $(0, 1)$, so that the ending and non-ending representations of x together give all the points in $(0, 1)$.

A number x in $(0, 1)$ being written in the form (3.1), let

$$(3.2) \quad y = F(x) = \frac{3^{a_1} q_1}{4} + \frac{3^{a_2} q_2}{4^2} + \dots + \frac{3^{a_n} q_n}{4^n} + \dots\dots$$

where, if k_v is a Φ being 0, 3, 4 or 7, q_v is 0, 3, 2, or 5 respectively, and if k_v is a ψ being 0, 1, 4 or 5, q_v is 0, 1, -2 or -1 respectively.

4. Taking PQ to be the join of (0, 0) and (1, 1), F_1 is the join of $(\frac{3}{8}, \frac{3}{4})$ and $(\frac{7}{8}, \frac{5}{4})$ to the middle point and the ends of PQ. Thus F_1 consists of four lines of which the first and third are positively inclined while the second and fourth are negatively inclined. On the negatively inclined stretches the construction is carried out as shown in Fig. 1, according to Kowalewski.

It will be observed that F_1 has three edge points (for $x = \frac{3}{8}, \frac{1}{2}, \frac{7}{8}$) besides the end points, P and Q, whose ordinates are not affected by subsequent constructions. From the construction given, it follows that the edge points of F_{n-1} are also the edge points of F_n, and further that the abscissae of the edge points of F_n are all the points x whose representations in the form (3.1) run up to n places. It is also easy to verify that the corresponding values of y obtained according to the analytical definition give the ordinates of the edge points of F_n.

The function Y and Bolzano's function, therefore, agree for an everywhere dense set of values of x in (0, 1), and as both can be shown to be continuous,* they must be identical.

The analytical definition given above provides an easy method of constructing a class of non-differentiable

*See Singh (80).

functions of Bolzano's type. To do this we have simply to choose proper values of the a's, k's and q's in (3.1) and (3.2). It may be further pointed out that the base of representation in (3.1) and (3.2) instead of being 8 and 4, may be any other properly chosen numbers.

5. **Koch's Curve.** In two papers, (45, 46) published during 1903-1906, Helge von Koch developed a new method of constructing plane curves having no determinate tangent at any point. The following example will illustrate Koch's construction.

Divide the straight line (A, B) by means of points C and E into three equal parts. Construct the equilateral triangle CDE on the middle part CE (see Fig. 2).

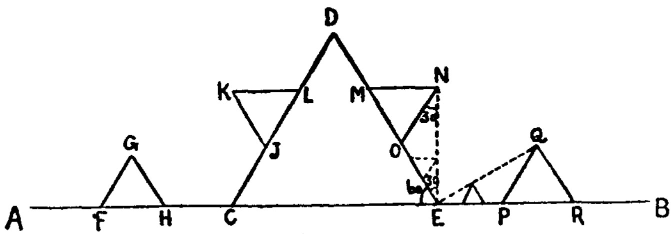


Fig. 2.

Apply the same construction to each of the four new lines AC, CD, DE and EB. Continue this construction indefinitely. The vertices of the equilateral triangles so obtained together with their limiting points form the curve of Koch.

It can be easily shown* that Koch's Curve corresponds to a multiple valued function $y=K(x)$. For

* See Singh (77).

example, the vertices of an indefinitely large number of triangles lie on the vertical through E (see Fig 2), and every vertical through a point (except the middle point) on CE cuts the curve more than once. The curve is, however, a parameter curve and can be expressed by equations of the form : †

$$x = \Phi(t)$$
$$y = \psi(t)$$

where $\Phi(t)$ and $\psi(t)$ are single valued and continuous functions of t.

It has been shown by F. Apt (3) that the multiple-valued curve $y = K(x)$ does not possess half-tangents.‡ But the function, being multiple-valued, can not be classed in the same category as Bolzano's curve or the functions defined by infinite series in the first lecture.

6. Parameter curves corresponding to multiple-valued functions have been defined by Peano, (59), Hilbert (35), Moore (57) Schoenflies (72), Sellerio (74), Kaufmann (39) and others. All these curves can be shown to be tangentless, and even without half-tangents at any point. Although the curves are not examples of continuous non-differentiable functions (being in a sense discontinuous because they are multiple-valued), yet for

†See Kaufmann ( ).

‡By a half-tangent at a point P is meant the limit of the secant PQ as Q approaches P always remaining on the same side of P.

the sake of their historical importance I shall give the definitions of some of them.

7. **Peano's Curve.** Peano, in 1890, defined a parameter curve $x=\Phi(t)$, $y=\psi(t)$, which passes through all the points of a unit square. The following geometrical method of obtaining the curve is based on the work of Moore ([57]) and Schoenflies ([72]).

Consider the diagonal AC of the unit square ABCD, and denote it by F_0. Divide the square into 3^2 equal parts and also the interval (0, 1) into 3^2 equal parts. Let the straight line AC be stretched and brought into the polygonal form shown in Fig. 3. Denote

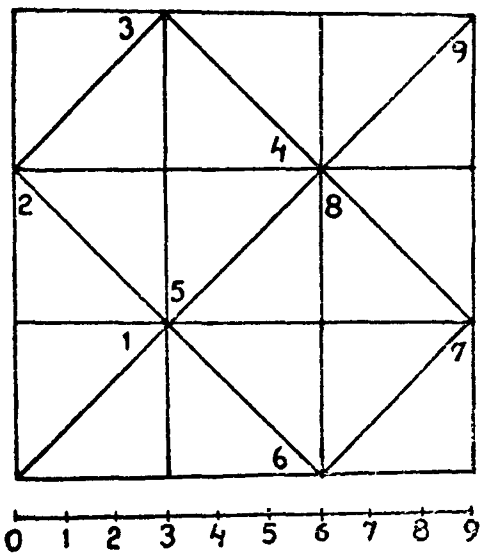


Fig. 3.

this polygonal stretch by F_1. In this manner the squares are arranged in the order 1, 2, 3,...3^2 and are placed into correspondence with the segments of (0, 1) bearing the same numbers. The stretch F_1 is made up of the diagonals of the small squares and is traversed from A to C in the order indicated by the numerals shown in the figure. The above construction is now applied to each diagonal of the small squares giving a polygonal stretch F_2, which now passes through each of the 3^4 squares into which the unit square is divided, and which goes from A to C. Continuing the construction we obtain polygonal stretches F_3, F_4...F_n...Peano's curve is $F = \lim\limits_{n \to \infty} F_n$.

We observe that the curve F_n passes through each of the 3^{2n} equal squares into which the unit square is divided. It follows that the curve $\lim\limits_{n \to \infty} F_n$ will pass through every point of the square *at least once*. Thus for any given value of x, the corresponding values of y are all the values in (0, 1). We also find that the secant line drawn from any given point on the curve can not converge to a fixed direction, in fact, it oscillates through 360^0 almost everywhere and *the curve has no half-tangent at any point.*

The associated (x, t) and (y, t) curves given by

$$x = \Phi(t) \text{ and } y = \Psi(t),$$

are each single-valued and continuous. The functions

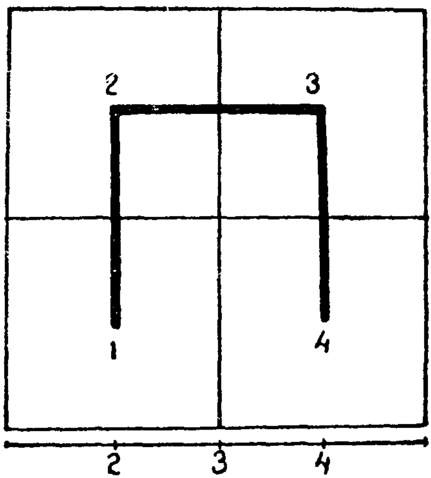
$\phi(t)$ and $\psi(t)$ have been shown to be non-differentiable functions by Moore ([57]) and Banerji ([4]). These functions will be considered in the next lecture. A different method of looking at the correspondence established by such curves will be illustrated by the following:

8. **Hilbert's Curve.** Let the variable t range over the interval I (0, 1) and let the point (x, y) range over the unit square R. Single-valued continuous functions

$$x = \phi(t),\ y = \psi(t)$$

can be defined, so that as t ranges over I, (x, y) ranges over the whole of the domain R. This can be done as follows:

Divide the interval I into four parts $\delta_1, \delta_2, \delta_3, \delta_4$, and the unit square R also into four parts, $\eta_1, \eta_2, \eta_3, \eta_4$.



Figs 4 *a*.

We call this the first division or D_1. The correspondence between I and R is given *in the first approximation* by saying that to each point P in δ, shall correspond some point Q in η.

Let the polygonal stretch shown in Fig. 4*a* be called F_1.

We now effect a second division D_2 by dividing the intervals and the squares of D_1 each into four equal parts. Let the numbering of these parts be carried out as shown in Fig. 4*b*.

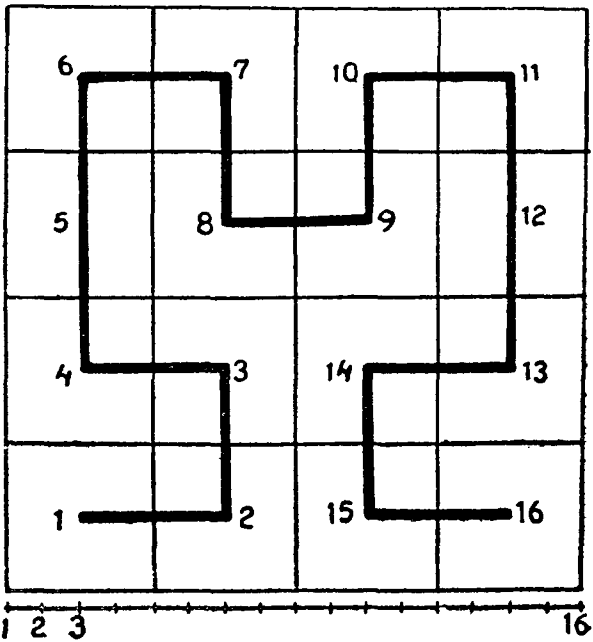


Fig. 4b.

Let the polygonal line shown in Fig. 4*b* be called F_2. The correspondence between I and R is given in the *second approximation* by saying that to a point P of the *r*th interval of I corresponds some point Q in the *r*th square of R.

The third division D_3 and the third polygonal stretch F_3 are illustrated in Fig. 4*c*.

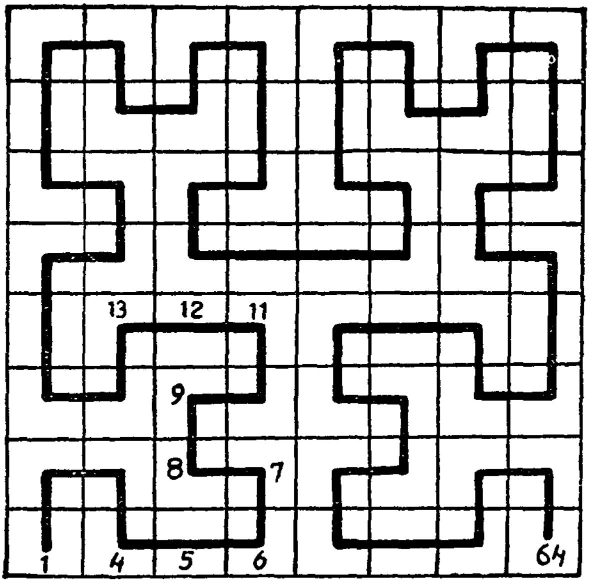


Fig. 4*c*.

The above construction is continued indefinitely. To find the point Q in R corresponding to P in I, we observe that P lies in a sequence of intervals tending to zero (in length), to which correspond uniquely a sequence of squares tending to zero (in area) and hence defining the point Q, whose co-ordinates are, therefore, single-valued functions of t.

The curves F_1, F_2, F_3......form a sequence of curves, and the limiting curve

$$F = \lim_{n \to \infty} F_n$$

is a curve which fills the unit square. It is obvious that the curve can not have a tangent or a half-tangent at any point.

The single-valued continuous functions

$$x = \Phi(t) \text{ and } y = \psi(t)$$

associated with Hilbert's curve have been defined analytically by R. D. Misra ([55]), who has also given a proof of their non-differentiability.

9. Space-filling Nature of the Curves.

The curves of Peano and Hilbert defined above fill the entire surface of a unit square. It has been pointed out by Moore that such curves can be obtained in an infinite variety of ways by assigning any suitable construction that is capable of systematic repitition indefinitely. Curves filling entirely a unit cube or n-dimensional space can also be constructed by a similar geometrical procedure. By a simple generalisation of Peano's method, Singh ([78]) has given an analytical method of obtaining curves which fill entirely a given n-dimensional space.

A curve filling the unit cube will be given analytically by the equations

$$x = \Phi_1(t), \quad y = \Phi_2(t), \quad z = \Phi_3(t);$$

where the point (x, y, z) ranges over the entire cube as t ranges over the linear interval $(0, 1)$. According to Hilbert a kinematical interpretation of the functional relation between (x, y, z) and t is that a point may move so that in unit time it passes through every point of the cube. This interpretation, however, can not be realized in practice, as the length of the (x, y, z) curve is infinite.

10. **Kaufmann's Curve.** The curve defined by Kaufmann ([39]) is a parameter curve of the same type as the curve of Koch. It corresponds to a multiple-valued function $y=f(x)$, and has at no point a half-tangent. The curve is defined as follows :

Inscribed polygonal stretches and the projection condition.—Let ABC be an isosceles triangle with the fundamental side AB and the base angle v. Let CM be the perpendicular from C on AB. Let $A_1, A_2, \ldots\ldots A_n \ldots\ldots$ be a sequence of different points on MA converging towards A, such that A_n lies between A and A_{n-1}. Correspondingly let $A'_1, A'_2 \ldots\ldots A'_n \ldots$ lying on CA be a sequence of different points converging towards A, and likewise so arranged that A'_n lies between A and A'_{n-1}. By joining these stretches CA_1, $A_1A'_1$, A'_1A_2, $A_2A'_2$

etc., we form a broken polygonal stretch ∇_{CA}. In the same way we define in the triangle BMC a polygonal stretch ∇_{CB}. The joint ∇ of the two polygonal stretches ∇_{CA} and ∇_{CB} we call a polygonal stretch inscribed in the $\triangle$ ABC.

We now keep in mind such polygonal stretches inscribed in the triangle ABC, all of whose stretches (suitably oriented) make a constant angle with the fundamental side AB. If, now, ν is the base angle of ABC, then, there is for every value of $\theta < \frac{\pi}{2} - \nu$ one and only one polygonal stretch inscribed to the $\triangle$ ABC all of whose stretches form with the fundamental side AB of the triangle ABC a fixed angle less than $\frac{\pi}{2} - \theta$ (see Fig. 5). We call ∇ in this case a polygonal stretch inscribed in the $\triangle$ ABC with the *reflection angle* θ. In the following we shall obtain, by iteration, with the help of such inscribed polygonal stretches those parameter curves which interest us.

Construction of the Curve.

We start from an isosceles $\triangle$ ABC with the base angle $\gamma < \frac{\pi}{6}$. By ∇° we denote a polygonal stretch inscribed in ABC with the constant reflection angle $\theta < \frac{\pi}{6}$. On each stretch $\top$ of ∇° inside the reflection space θ, we construct isosceles triangles with the fundamental side T and base angle less than θ. We obtain in this manner a sequence of triangles which we denote as chain triangles of the first order. Let, again, an inscribed polygonal stretch with the reflection angle θ be given in each chain triangle of the first order. Such a polygonal stretch may be likewise called of the first order. We choose each stretch T of everyone of the polygonal stretches of the first order as a fundamental side of an isosceles triangle (inscribed in the angular space θ) *with the base angle* γ. Every one of such

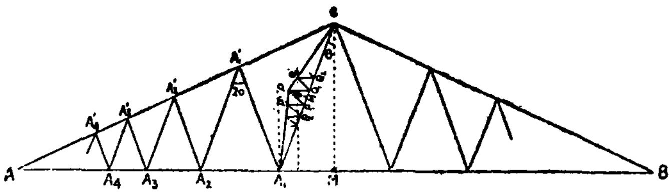


Fig. 5.

triangles we call a chain-triangle of the second order, and we determine in this an inscribed polygonal stretch of the second order with the *reflection angle* θ, and so on. In such a manner will be defined chain-triangles of the nth order ($n=1, 2,...$) and corresponding inscribed polygonal stretches of the nth order. The joining together of all the inscribed polygonal stretches of any fixed order n gives a simple curvilinear arc. The sequence of these curvilinear arcs for increasing n converges as is easily seen towards a simple curve l.

11. **Kaufmann's curve is multiple-valued.** It is easy to show that Kaufmann's curve l, defined as above, corresponds to a multiple-valued function $y=f(x)$. Let $A_1P\,C$ be the triangle constructed on the first stretch CA_1 (see fig. 5). Let PP_1, $P_1P'_1$, P'_1P_2........, and PQ_1, $Q_1Q'_1$, Q'_1Q_2,......be the parts of the polygonal stretch inscribed in the triangle A_1PC. As the stretches PQ_1 and PP_1 each make an angle θ with PM_1, it is easy to see that PQ_1 is horizontal whilst PP_1 makes an angle 2θ with it. Now, as $2\theta < \frac{\pi}{3}$, therefore, the vertical drawn through any point on PP_1 must also cut the stretch PQ_1. This vertical, therefore, cuts the limiting curve l in at least two points lying on the two portions of l which correspond to the stretches PP_1 and PQ_1. In fact, the vertical will *in general* cut the curve l in an infinite number of points. The function $y=f(x)$ is, therefore, a multiple-valued function.

12. **Non-existence of half-tangents.** Kaufmann has given an indirect proof of the non-existence of half tangents at any point of the curve. The property, however, is an immediate corollary from the multiple-valued nature of the curve at an everywhere dense set of values of x. For, let (x, y) be any point on the curve and let $x_1, x_2, \ldots x_n, \ldots$ be a sequence approaching x from the right, and let $y_1, y_2 \ldots y_n \ldots$ and $y'_1, y'_2 \ldots y'_n, \ldots$ be the corresponding values of y, lying on different parts of the

curve. It is easy to see that the angle between the two secants joining (x, y) with (x_n, y_n), and (x, y) with (x'_n, y'_n) always make with each other an angle greater than θ, so there can be no limiting secant on the right at any point. Similarly on the left there can be no limiting secant. *This shows the futility of considering parameter curves as examples of non-differentiable functions.*

13. **Modifications of Kaufmann's curve.** A single-valued function can be obtained by a suitable modification of Kaufmann's construction. It will be shown that the single-valued function, so obtained, possesses half-tangents at an everywhere dense set of points, and is differentiable there.

Let PA_1C be the first chain-triangle lying in the part AMC of the triangle ABC, and let the angle $CA_1M = a_0$.

The base angle ν_1 of the triangle PA_1C is so chosen that the inclination of each of the sides PA_1 and PC to the horizontal is less than $\frac{\pi}{2}$. Through P draw the vertical PN_1 (see fig. 6). We have now to construct a series of stretches in the triangle PA_1C, in such a way that the resulting curve, represented by these stretches shall be single-valued. For this purpose we draw through P a straight line PP_1 cutting A_1C in P_1 and lying in the angular space A_1PN_1. Let a_1 be the inclination of PP_1

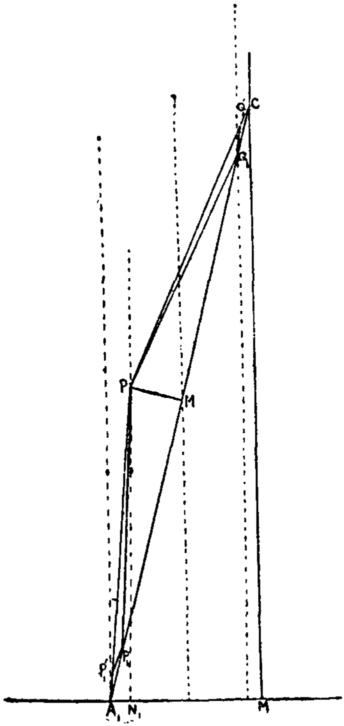


Fig 6.

to the horizontal. The other stretches $P_1P'_1, P'_1P_2 \ldots$ and PQ_1, Q_1Q_1', $Q'_1Q_2 \ldots$ are all constructed so that they are equally inclined to A_1C just as in Kaufmann's construction. Similar stretches are constructed in all the chain triangles. This gives us the stretches of the first order. On each stretch we have now to construct an isosceles triangle. For fixity of ideas we consider the stretch PP_1. Construct on it an isosceles triangle

P^1P_1P with the base angle ν_2, such that the inclination of each side P^1P_1 and P^1P is less than $\frac{\pi}{2}$. Similar triangles are constructed on all the stretches. In each chain-triangle of the second order so obtained we shall have stretches of the second order. $P^1P_1{}^1$ is the first stretch lying on the left of the triangle P^1P_1P, so constructed that it lies in the angular space between P^1P_1 and the vertical through P^1. The other stretches are then constructed as before. This procedure gives, at the nth stage, a curve l_n which is made up of the stretches of the nth order. The curve $l = \lim\limits_{n\to\infty} l_n$ corresponds to a single valued function, $y=f(x)$, as is easily seen.

14. **The existence of a derivative.** Let the first chain-triangle lying in the left part of the $\triangle$ ABC be denoted by $\triangle_{0,1}$, and let its sides CA_1, A_1P and PC be denoted by $S_{0,1}$, $S_{0,2}$, $S_{0,3}$. Let the first chain-triangle constructed in $\triangle_{0,1}$ and lying in the left-part (lower part) of $\triangle_{0,1}$ be denoted by $\triangle_{1,1}$ and its corresponding sides by $S_{1,1}$, $S_{1,2}$, $S_{1,3}$. We have likewise $\triangle_{n,1}$, and its corresponding sides $S_{n,1}$, $S_{n,2}$, $S_{n,3}$. We also know that $S_{0,1}$, $S_{1,1}$,......$S_{n,1}$,...... belong to the stretches of different orders $(0, 1, 2,...n,...)$. Their inclinations to the horizon a_0, a_1, a_2,...a_n,...form a monotone increasing sequence, and as each is less than $\frac{\ \ }{2}$

$$\lim_{n\to\infty} a_n = a \leq \frac{\pi}{2} \qquad \text{.......................}(1)$$

Let $\nu_1, \nu_2, \nu_3, \ldots \nu_n, \ldots$ be the base angles of $\triangle_{0,1}, \triangle_{1,1}, \ldots \triangle_{n,1}, \ldots$

It is easy to see that because of (1)

$$\lim_{n\to\infty} \nu_n = 0 \qquad \text{..............................}(2)$$

Moreover the inclinations of $S_{n,1}$, $S_{n,2}$ and $S_{n,3}$ are given by a_n, $a_n+\nu_n$ and $a_n-\nu_n$. It follows, therefore, that $S_{n,1}$, $S_{n,2}$ and $S_{n,3}$ each have the same limiting inclination a as n tends to infinity.

Consider now the point S which belongs to all the chain triangles $\triangle_{0,1}, \triangle_{1,1}, \ldots \triangle_{n,1}, \ldots$ This point is the limiting point of the vertices of the chain triangles, as n tends to infinity. Moreover, it lies in the left-part (lower part) of each chain triangle of the sequence given above. It is further easy to see that the inclination of any secant joining the point S to any point of the curve l corresponding to the side $S_{n,3}$ of $\triangle_{n,1}$ lies between a_n and a. And as

$$\lim_{n\to\infty} a_n = a,$$

therefore, the secants drawn from S to points of the curve l tend to the limiting inclination a. Thus at the point S, the curve has a progressive derivative $= \tan a$, where $a \leq \frac{\pi}{2}$.

It can be similarly proved that at S a regressive derivative $= \tan a$ exists.

Thus at the point S, there exists a differential coefficient whose value is equal to tan a; $a \leq \frac{\pi}{2}$.

The above proof can be evidently applied to show that *there exists an everywhere dense set of points at each of which the function possesses a differential coefficient.*

15. **Besicovitch's Curve.** I shall now give the construction of the Curve of Besicovitch, which has been stated to be without half-tangents. The curve corresponds to a single-valued function and is defined as follows:

" Let us take the stretch $AB = 2a$ for A $(0, 0)$ and $B(2a, 0)$, and the points $C(a, b)$ and $D(a, 0)$. On the stretch AD let us construct a stretch $l_1 = \frac{a}{4}$ placing it centrally. The stretch AD is divided by the stretch l_1 into two equal parts. On each of these let us place centrally the stretches $l_2 = l_3 = \frac{a}{2^4}$. The stretches l_1, l_2, l_3 divide the stretch AD into four equal stretches. On each of these let us place centrally (calculated from left to right) the stretches $l_4 = l_5 = l_6 = l_7 = \frac{a}{2^6}$, and so on. In this manner a set L of stretches

$$l_1 + l_2 + l_3 + \cdots = \frac{a}{2}$$

is constructed on the stretch AD.

We construct a similar system of stretches on DB. We call these stretches the first series of stretches.

Let us denote by $m(x)$ the measure* of the set of points of the interval $(0, x)$ which do not belong to the set L, and let us determine on the stretch AD a function $\varphi(x)$, whilst we assume

$$\varphi(x)=\frac{2b}{a}m(x).$$

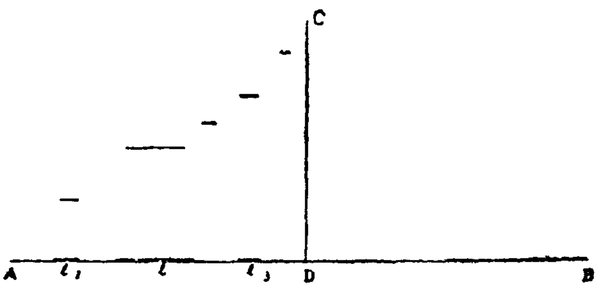


Fig 7.

The points A and D are thus connected by the curve $y=\varphi(x)$, which has a constant value on an arbitrary stretch l_r and which we call a 'ladder curve'; the points C and B are likewise connected by such a ladder curve. The figure originating in this manner is called a 'step-triangle' whose base is $2a$ and whose height is b (see Fig. 7).

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* It has been assumed that the measure $m(x)$ exists as a unique number for every x.

On the fundamental lines corresponding to the first series of stretches of the step-triangle ABC, let us construct step-triangles directed towards below, equal on equal fundamental lines, whilst we choose the height so that the vertex of the undermost of all equal triangles lies on the side AB. The construction of all these triangles is called the operation of ' maiming ' the triangle ABC towards inside. With the so obtained infinity of triangles (first series) we carry out the same operation of maiming towards inside, and thus obtain the second series of triangles; on them also perform maiming towards inside, and so on.

We now define a function $f(x)$ on the stretch AB as follows:

(1) at the points of the stretch AB, which do not belong to the first series of stretches, by the ordinates of the sides of the step-triangle ABC;

(2) at the points of the stretches of the first series, which do not belong to the stretches of the second series, by the ordinates of the sides of the triangles of the first series;

(3) at the points of the stretches of the second series, which do not belong to the stretches of the third series, by the ordinates of the sides of the triangles of the second series, and so on;

(4) at the points which belong to the stretches of all series (they form a null set) according to the principle of continuity."

It has been stated by Besicovitch ([5]) and E. D. Pepper ([62]) that the curve does not possess a half-tangent (progressive or regressive) at any point. Whilst it is certain that the curve is single-valued and non-differentiable, there is considerable doubt regarding the validity of the proof, given by Besicovitch and Pepper, for the non-existence of half-tangents.* The proof given by these writers depends upon geometrical intuition whose validity can not be defended. If an arithmetic definition of the function could be devised, it would be possible to study the derivates of Besicovitch's function in detail and to decide whether it has progressive derivatives or not.

16. **Other Curves.** In a series of papers published during 1907—1910, Faber ([25, 26, 27]) gave a geometrical method for the construction of non-differentiable functions. Faber's functions are single-valued and continuous, and have the additional advantage of being capable of expression as infinite series. It has already been pointed out that these functions are special cases of the general class considered by Knopp ([44]). Landsberg ([51]) has also constructed non-differentiable functions, which like those of Faber, are obtained by geometrical construction in a simple manner.

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* See Singh (82).

THIRD LECTURE

FUNCTIONS DEFINED ARITHMETICALLY

1. **Introduction.** In the year 1890, Peano [59] defined a surface-filling curve by the help of arithmetically defined functions

$$x = \Phi(t) \text{ and } y = \Psi(t).$$

The function $\Phi(t)$ is defined as follows:

Let a point t of the interval $(0, 1)$ be represented arithmetically in radix fractions in the scale of 3 as

$$t = \frac{a_1}{3} + \frac{a_2}{3^2} + \frac{a_3}{3^3} + \cdots\cdots + \frac{a_n}{3^n} + \cdots\cdots\cdots,$$

where the a's are 0, 1 or 2.

Corresponding to t let a number $x = \Phi(t)$ be defined as follows:

$$x = \Phi(t) = \frac{b_1}{3} + \frac{b_2}{3^2} + \cdots\cdots + \frac{b_n}{3^n} + \cdots\cdots\cdots\cdots$$

where $b_1 = a_1$, and $b_n = \mathrm{K}^{a_2 + a_4 + \cdots + a_{2n-2}}(a_{2n-1})$,

and $\mathrm{K}^p(a) = a$ or $(2 - a)$ according as p is even or odd.

Similarly, the function $\Psi(t)$ is defined as:

$$y = \Psi(t) = \frac{c_1}{3} + \frac{c_2}{3^2} + \cdots\cdots\cdots + \frac{c_n}{3^n} + \cdots\cdots$$

where $c_n = \mathrm{K}^{a_1 + a_3 + \cdots + a_{2n-1}}(a_{2n})$, $(n = 1, 2, \ldots\ldots)$,

and the operator K has the same meaning as before.

Peano's functions were generalized, and a proof of their non-differentiability was published by E. H.

Moore([57]) in 1900. Moore's treatment is geometrical. An analytical proof of the non-differentiability of $\Phi(t)$ has been given by H. P. Banerji ([4]). In to-day's lecture I shall consider a class of functions of which Peano's and Moore's functions are special cases.

A general theory of the construction of non-differentiable functions was published by Steinitz ([86]) in 1899. He divides the interval (a, b) into m equal parts and prescribes the 'value-difference' of a function $\Phi(x)$ for each of these m parts. He then divides each part again into m equal parts, and prescribes the 'value-differences' for each of the new m^2 parts. Proceeding in this manner he obtains a function $\Phi(x)$ valid for an everywhere dense set in (a, b). By the extension of $\Phi(x)$ he obtains a function $f(x)$ defined for the whole interval (a, b), but fails to give a sufficient condition for the non-existence of finite or infinite differential coefficients. He has, however, indicated a method of obtaining a non-differentiable function when m is greater than 5.

In a paper, published in 1918, H. Hahn ([31]) has considered a function, constructed according to the method of Steinitz (for $m=6$), and proves that at no point does the function possess both the progressive and regressive differential coefficients.

The advantage of the arithmetic definition over all other forms of definitions is that the numerical value of the function at each point is directly given by the definition so that the character of the derivates at any assigned point can be studied directly. That arithmetically defined functions have been attracting attention is shown by the attempts of several mathematicians to construct such functions. As early as 1904, T. Takagi [90] constructed an arithmetically defined function by using the representation of the points of a linear interval in the scale of 2. Takagi's function is non-differentiable at an everywhere dense set of points. E. Cesaro [19], in 1905, gave an arithmetically defined function which has no differential coefficient at the points of an everywhere dense set. By representing the points x of a linear interval in the scale of 5, W. Sierpinski [76], in 1914, obtained a function $y=f(x)$, represented in the scale of 3, which does not simultaneously possess both progressive and regressive differential coefficients at any point. A class of simple non-differentiable functions was given by K. Petr [60] in 1920. By using the representation of the points x of a linear interval in an *even* scale 2b, Petr obtained a function $y=f(x)$ expressed in another *even* scale $2c$ $(b>c)$.

It will be observed that radix fractions have been used for obtaining the definitions of the above functions. But the use of radix functions is not necessary as is shown by the definition of Bolzano's function given in the second lecture. In fact all that is necessary is to set up a correspondence between the points x of an everywhere dense set in an interval, say (0, 1), with those of another set y, in such a way that, (i) to every value of x there corresponds only one value of y, (ii) to every value of y, there corresponds an infinite number of values of x, (iii) to the two representations of x (ending and non-ending) there corresponds the same value of y, and (iv) the ratio $\frac{y-y'}{x-x'}$ tends to infinity as x' tends to x.

Cesaro (18) has pointed out that Koch's curve can be defined arithmetically. An arithmetic definition of Hilbert's curve has been given by R. D. Misra (55). The curves given in the second lecture can also be defined arithmetically.

I shall now give a detailed study of the derivates of a class of non-differentiable functions constructed by me. I believe that in the case of no other function have the derivates been so completely studied.

2. Definition of the functions $\phi_{m,r,p}(x)$.

Let the numbers in the interval (0, 1) be expressed in radix fractions, in the base p, where p is an odd integer.

Then a point x in (0, 1) can be represented as

$$x=\frac{a_1}{p}+\frac{a_2}{p^2}+\frac{a_3}{p^3}+\cdots+\frac{a_n}{p^n}+\cdots.$$

where the a's are positive integers such that

$$0 \leq a \leq p-1.$$

For a given integer r and another given integer m, let

$$Y_{m,r,p}=\Phi_{m,r,p}(x) = \frac{b_{1,r}}{p}+\frac{b_{2,r}}{p^2}+\cdots+\frac{b_{n,r}}{p^n}+\cdots.$$

where

$$b_{1,r}=k^{a_1+a_2+\cdots+a_{r-1}}(a_r),$$

$$b_{2,r}=k^{a_1+a_2+\cdots+a_{r-1}+\cdots a_{m+r-1}}(a_{m+r}),$$

$$b_{3,r}=k^{a_1+\cdots+a_{r+1}+\cdots+a_{m+r+1}+\cdots+a_{2m+r-1}}(a_{2m+r}),$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

and so on; where $k^s(a)=a$ or $(p-1-a)$ according as s is even or odd.

In the above, the integer p may have any odd value $3, 5, 7, \ldots$, the integer m may have any one of the values $2, 3, 4, \ldots$ and r is an integer $\leq m$. The function $\Phi_{2,1,3}(x)$ is Peano's function $\phi(t)$, and the function $\Phi_{2,2,3}(x)$ is Peano's function $\psi(t)$; while the functions $\Phi_{2,1,p}(x)$ and $\Phi_{2,2,p}(x)$, where p is an odd integer ≥ 3, are the functions considered by Moore.

3. **The function** $\Phi_{3,1,3}(x)$. I give below a proof of the continuity and the non-differentiability of the function $\Phi_{3,1,3}(x)$. The proof for the general case can be carried out exactly in the same manner.

We have, if

$$x=\frac{a_1}{3}+\frac{a_2}{3^2}+\cdots\cdots+\frac{a_n}{3^n}+\cdots\cdots$$

$$Y_{3,1,3}=\Phi_{3,1,3}(x)$$

$$= \frac{b_{1,1}}{3} + \frac{b_{2,1}}{3^2} + \cdots\cdots + \frac{b_{n,1}}{3^n} + \cdots\cdots$$

which, dropping out the suffixes, we write as

$$y = \Phi(x) = \frac{b_1}{3} + \frac{b_2}{3^2} + \cdots\cdots + \frac{b_n}{3^n} + \cdots\cdots,$$

where

$$b_1 = k^0(a_1) = a_1$$

$$b_2 = k^{a_2+a_3}(a_4)$$

$$b_3 = k^{a_2+a_3+a_5+a_6}(a_7)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$b_n = k^{a_2+a_3+a_5+a_6+\cdots+a_{3(n-1)-1}+a_{3(n-1)}}(a_{3(n-1)+1})$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

and so on; and where $k^s(a) = a$ or $(2-a)$ according as s is even or odd.

4. $Y = \Phi(x)$ is a continuous function of x. The points x which have a non-ending representation such that all the a's, from and after a_n are not all 2's, are uniquely representated in a radix fraction, and thus for all such points we have a single value of Y.

When x is representable as an ending radix fraction, it has also a non-ending representation in which all the a's, from and after some place, are equal to 2.

Three cases arise:

(1) $x_1 = \dfrac{a_1}{3} + \dfrac{a_2}{3^2} + \cdots\cdots + \dfrac{a_{3n}}{3^{3n}}$,

$$= \frac{a_1}{3} + \frac{a_2}{3^2} + \cdots + \frac{(a_{3n}-1)}{3^{3n}} + \frac{2}{3^{3n+1}}$$

$$+ \frac{2}{3^{3n+2}} + \cdots\cdots$$

$$(2) \quad x_2 = \frac{a_1}{3} + \frac{a_2}{3^2} + \cdots + \frac{a_{3n}}{3^{3n}} + \frac{a_{3n+1}}{3^{3n+1}}.$$

$$= \frac{a_1}{3} + \frac{a_2}{3^2} + \cdots + \frac{(a_{3n+1}-1)}{3^{3n+1}} + \frac{2}{3^{3n+2}}$$

$$+ \frac{2}{3^{3n+3}} + \cdots\cdots$$

$$(3) \quad x_3 = \frac{a_1}{3} + \frac{a_2}{3^2} + \cdots + \frac{a_{3n+2}}{3^{3n+2}}.$$

$$= \frac{a_1}{3} + \frac{a_2}{3^2} + \cdots + \frac{(a_{3n+2}-1)}{3^{3n+2}} + \frac{2}{3^{3n+3}}$$

$$+ \frac{2}{3^{3n+4}} + \cdots\cdots$$

Case (1). Let the values of Y corresponding to the two representations of x be

$$Y_1 = \frac{b_1}{3} + \frac{b_2}{3^2} + \cdots + \frac{b_n}{3^n} + \frac{b_{n+1}}{3^{n+1}} + \cdots\cdots$$

and

$$Y'_1 = \frac{b'_1}{3} + \frac{b'_2}{3^2} + \cdots + \frac{b'_n}{3^n} + \frac{b'_{n+1}}{3^{n+1}} + \cdots\cdots$$

respectively.

Then as the two representations of x agree up to $3n$ places,

$$b_1 = b'_1,\ b_2 = b'_2,\ \ldots,\ b_n = b'_n,$$

and we further have

$$b_{n+1}=k^{a_2+a_3+\cdots+a_{3n-1}+a_{3n}}(0),$$
$$b_{n'+1}=k^{a_2+a_3+\cdots+a_{3n-1}+(a_{3n}-1)}(2)$$

and in general

$$b_{n+l}=k^{a_2+a_3+\cdots+a_{3n-1}+a_{3n}}(0)$$
$$b'_{n+l}=k^{a_2+a_3+\cdots+a_{3n-1}+(a_{3n}-1)+4(l-1)}(2)$$

for $l=1, 2, 3, \ldots\ldots$

We see that if the index of k in the first case (*i.e.* b_{n+l}) is odd, it is even in the second case (*i.e.* b'_{n+l}); and if it is even in the first case, it is odd in the second.

Thus

$$b_{n+1}=b'_{n+1},\ b_{n+2}=b'_{n+2},\ \ldots,\ b_{n+l}=b'_{n+l}\ \ldots\ldots$$

$\therefore Y_1=Y'_1$, and Y is uniquely determined for the points of *Case* (1).

Case (2). Let the two representations of Y corresponding to the two representations of x be

$$Y_2=\frac{b_1}{3}+\frac{b_2}{3^2}+\cdots+\frac{b_n}{3^n}+\frac{b_{n+1}}{3^{n+1}}+\cdots\cdots$$

and

$$Y'_2=\frac{b'_1}{3}+\frac{b'_2}{3^2}+\cdots+\frac{b'_n}{3^n}+\frac{b'_{n+1}}{3^{n+1}}+\cdots\cdots$$

We have as before

$$b_1=b'_1,\ b_2=b'_2,\ \ldots\ldots\ b_n=b'_n.$$

Now let

$$a_2+a_3+a_5+a_6+\cdots\cdots+a_{3n-1}+a_{3n}=\lambda$$

Then

$$b_{n+1}=k^{\lambda}(a_{3n+1}),\ b'_{n+1}=k^{\lambda}(a_{3n+1}-1)$$

and in general

$$b_{n+l}=k^{\lambda}(0),\ b'_{n+l}=k^{\lambda+4(l-1)}(2)$$

(for $l=2, 3, 4, \ldots\ldots$).

Let λ be even, then

$$b_{n+1}=a_{3n+1} \text{ and } b'_{n+1}=(a_{3n+1}-1)$$

while $b_{n+l}=0$, and $b'_{n+l}=2$ (for $l=2, 3, \ldots$).

$$\therefore Y_2=\frac{b_1}{3}+\frac{b_2}{3^2}+\cdots+\frac{b_n}{3^n}+\frac{a_{3n+1}}{3^{n+1}}$$

$$\text{and}\quad Y'_2=\frac{b_1}{3}+\frac{b_2}{3^2}+\cdots+\frac{b_n}{3^n}+\frac{(a_{3n+1}-1)}{3^{n+1}}+\frac{2}{3^{n+2}}$$

$$+\frac{2}{3^{n+3}}+\cdots$$

Hence $Y_2=Y'_2$.

Similarly, if λ be odd, it can be proved that

$$Y_2=Y'_2.$$

Therefore, Y is uniquely determined for the points of *Case* (2).

Case (3). Proceeding as in *Case* (1) we can show that Y is uniquely determined for the points of *Case* (3).

Thus Y is a single-valued function.

We observe that if two points x and x' agree up to the first $3n$ places in their representations, the corresponding values of Y agree up to the first n places.

Therefore, Y is a continuous function of x in $(0, 1)$.

5. $Y=\Phi(x)$ is nowhere differentiable. Non-existence of cusps.

For the purpose of proving the non-differentiability of $\Phi(x)$ we divide the numbers of the interval (0, 1) into the following four classes.

Class I (a). A number belongs to this class, if in its representation $(a_{3n-1}+a_{3n}) \leq 2$ and a_{3n+1} is not equal to 1, for an infinite number of values of n.

Class I (b). A number belongs to this class, if in its representation $(a_{3n-1}+a_{3n}) \leq 2$ and $a_{3n+1}=1$ for an infinite number of values of n.

Class II (a). A number belongs to this class, if in its representation $(a_{3n-1}+a_{3n}) > 2$ and a_{3n+1} is not equal to 1, for an infinite number of values of n.

Class II (b). A number belongs to this class, if in its representation $(a_{3n-1}+a_{3n}) > 2$ and $a_{3n+1}=1$ for an infinite number of values of n.

It is easy to see that a given number in (0, 1) must belong to at least* one of these four classes.

Class I(a). Let $h_n=\dfrac{2}{3^{3n}}$[n having a value for which a_{3n-1}, a_{3n} and a_{3n+1} satisfy the conditions of I (a)].

Then

$$\Phi(x)=\Phi(x+h_n),$$

* A number may evidently belong to more than one class.

$$\therefore \lim_{h_n=0} \frac{\Phi(x+h_n)-\Phi(x)}{h_n}=0.$$

Again, let $h'_n=\frac{1}{3^{3n}}$, n having the same value as above, then

$$\Phi(x+h'_n)-\Phi(x) \,|\, > \frac{1}{3^{n+1}}$$

$$\therefore \lim_{h'_n=0} \left| \frac{\Phi\,(x+h'_n)-\Phi(x)}{h'_n} \right| > \lim_{n=\infty} \frac{3^{3n}}{3^{n+1}}$$
$$= \infty.$$

$\therefore$ The progressive derivative does not exist at the points belonging to *Class* I (a).

Class I (b). Taking $h_n=\frac{2}{3^{3n}}$, we see as before that

$$\lim_{h_n=0} \frac{\Phi(x+h_n)-\Phi(x)}{h_n}=0.$$

Taking $h'_n=\frac{1}{3^{3n+1}}$, we see that

$$\left| \Phi(x+h'_n)-\Phi(x) \right| = \frac{1}{3^{n+1}}$$

$$\therefore \lim_{h'_n=0} \left| \frac{\Phi\,(x+h'_n)-\Phi\,(x)}{h'_n} \right| = \infty.$$

$\therefore$ The progressive derivative does not exist at the points belonging to *Class* I (b).

Class II (a). Taking h_n and h'_n as in I (a), we see that

$$\lim_{h_n=0} \frac{\Phi(x-h_n)-\Phi(x)}{-h_n} = 0$$

and
$$\lim_{h'_n=0} \left| \frac{\Phi(x-h'_n)-\Phi(x)}{-h'_n} \right| = \infty .$$

$\therefore$ The regressive derivative does not exist at the points belonging to *Class* II (a).

Class II (b). Taking h_n and h'_n as in I(b), we see that
$$\lim_{h_n=0} \frac{\Phi(x-h_n)-\Phi(x)}{-h_n} = 0$$

and
$$\lim_{h'_n=0} \left| \frac{\Phi(x-h'_n)=\Phi(x)}{-h'_n} \right| = \infty .$$

$\therefore$ The regressive derivative does not exist at the points belonging to class II (b).

We have thus shown that at the points belonging to *Class* I, the progressive derivative does not exist, while at the points belonging to *Class* II, the regressive derivative does not exist. As any point in (0, 1) must belong to at least one of these classes, therefore, at no point x in (0, 1) do both the derivatives exist, *i.e.* $\Phi(x)$ is nowhere differentiable in (0, 1). Also *the function* $\Phi(x)$ *does not possess a cusp at any point* [*].

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[*] *Cf.* Hobsou; *Theory of functions*, etc., Vol. II. (1926) p., 405, where it is stated, "It does not appear to be definitely known whether a non-differentiable function can exist which has no cusps.' ***The analysis of the derivates given above definitely answers this question.***

6. Existence of the derivative of $\phi(x)$.

(6.1) It has already been shown that at the points belonging to *Class* I, $\Phi(x)$ does not possess a right hand derivative; hence, in order to get the set of those points where $\Phi(x)$ possesses a right hand derivative, we search among those points x in whose representation $(a_{3n-1}+a_{3n})>2$ from and after some value v of n. We can easily show that if, for a point x, $(a_{3n-1}+a_{3n})=3$ for an infinite number of values of n, then the right hand derivative is non-existent at x. Thus the points where a right hand derivative may exist are such that in their representation $(a_{3n-1}+a_{3n})=4$ for all values of n greater than or equal to some number v.

(6.2) Let x be such that in its representation $(a_{3n-1}+a_{3n})=4$ for all values of $n \geq v$ and a_{3n+1} is either 0 or 1 for all values of $n \geq v$. Further let

$$\sum_{1}^{v-1}(a_{3r-1}+a_{3r}) \text{ be even.}$$

Then a point x of this type has the representation

$$(6.21)\quad x=\frac{a_1}{3}+\frac{a_2}{3^2}+\cdots+\frac{2}{3^{3v-1}}+\frac{2}{3^{3v}}+\frac{a_{3v+1}}{3^{3v+1}}$$
$$+\frac{2}{3^{3v+2}}+\frac{2}{3^{3v+3}}+\frac{a_{3v+4}}{3^{3v+4}}+\cdots+\frac{2}{3^{3m-1}}+\frac{2}{3^{3m}}$$
$$+\frac{a_{3m+1}}{3^{3m+1}}+\cdots$$

A point x' lying on the right of x sufficiently near it, must have the representation

$$(6.22)\quad x' = \frac{a_1}{3} + \cdots + \frac{a_{3v+1}}{3^{3v+1}} + \cdots + \frac{2}{3^{3m-1}} + \frac{2}{3^{3m}}$$

$$+ \frac{a'_{3m+1}}{3^{3m+1}} + \frac{a'_{3m+2}}{3^{3m+2}} + \frac{a'_{3m+3}}{3^{3m+3}} + \cdots\cdots,$$

where

$$(6.23)\qquad a'_{3m+1} > a_{3m+1}$$

The corresponding representations of $\Phi(x)$ and $\Phi(x')$ agree up to m places. After the mth place the representations are: for $\Phi(x)$

$$(6.24)\qquad \frac{a_{3m+1}}{3^{m+1}} + \frac{a_{3(m+1)+1}}{3^{m+2}} + \cdots + \frac{a_{3l+1}}{3^{l+1}} + \cdots\cdots,$$

and for $\Phi(x')$

$$(6.25)\qquad \frac{a'_{3m+1}}{3^{m+1}} + \frac{b_{m+2}}{3^{m+2}} + \cdots + \frac{b_{l+1}}{3^{l+1}} + \cdots\cdots$$

Therefore,

$$(6.26)\qquad \Phi(x') - \Phi(x) = \frac{(a'_{3m+1} - a_{3m+1})}{3^{m+1}}$$

$$+ \frac{(b_{m+2} - a_{3(m+1)+1})}{3^{m+2}} + \cdots\cdots + \frac{(b_{l+1} - a_{3l+1})}{3^{l+1}} + \cdots\cdots$$

Now, the least value that the b's can have is *zero*, while $(a'_{3m+1} - a_{3m+1}) \geq 1$, by (6.23).

$$(6.27)\qquad \therefore\ \Phi(x') - \Phi(x) \geq \frac{1}{3^{m+1}} - \left\{ \frac{1}{3^{m+2}} \right.$$

$$\left. + \frac{1}{3^{m+3}} + \cdots\cdots \right\} \geq \frac{1}{2}\,\frac{1}{3^{m+1}}.$$

for the a_{3r+1}'s are either 0 or 1.

(6.28) Also $$(x'-x) \leq \frac{1}{3^{3m}}.$$

Therefore,

$$\underset{x'=x}{Lim} \frac{\Phi(x')-\Phi(x)}{x'-x} > \underset{m=\infty}{Lim} \frac{\frac{1}{2} \cdot \frac{1}{3^{m+1}}}{\frac{1}{3^{3m}}}$$

$$> \underset{m=\infty}{Lim} \tfrac{1}{2} 3^{2m-1} = \infty.$$

Therefore, at the points of this type, there exists a right hand derivative equal to ∞.

(6.3) Taking the a's up to a_{3v-2} to be fixed, the points of the type (6.2) (for this fixed v) are obtained by giving to the $a_{3(v+r)+1}$'s the values 0 or 1 (where $r=0,1,2,3,\ldots$). Thus these numbers can be placed into one to one correspondence with the numbers of the continuum (0, 1) expressed as radix fractions in the scale of 2. The cardinal number of these points is, therefore, c. As v can have all finite integral values, the set of all the points of the type (6.2) is everywhere dense in (0,1).* Moreover, the set of points of the type (6.2), for a fixed v, form a non-dense set. Giving to v the values 1, 2, 3,...we get a sequence of such sets. It follows, therefore, that the set of all the points of the type (6.2) form an unenumerable, everywhere dense set of the first category in (0, 1).

* For, given any interval (α,β) in (0, 1), we can easily find a point of the type (6. 2) in (α, β). To do this we have simply to choose the a's properly.

(6.4) Let $(a_{3n-1}+a_{3n})=4$ for all values of $n \geq v$. If μ_m denote the number of a_{3n+1}'s which are equal to 2, $n=m+1, m+2 \ldots\ldots\ldots$, immediately following a_{3m+1}(not equal to 2), let

$$\underset{m=\infty}{Lim} \ (2m-\mu_m)=\infty,$$

and further, let $\sum_{1}^{v-1} (a_{3n-1}+a_{3n})$ be even.

Then reasoning as before, corresponding to the inequality (6.27), we get

$$\Phi(x')-\Phi(x) \geq \frac{1}{3^{m+1}} - \left\{ \frac{2}{3^{m+2}} + \frac{2}{3^{m+3}} + \cdots + \frac{2}{3^{m+\mu_m+2}} \right\}$$

$$\geq \frac{1}{3^{m+1}} - \frac{2}{3^{m+2}} \left\{ 1 + \frac{1}{3} + \cdots + \frac{1}{3^{\mu_m}} \right\}$$

$$\geq \frac{1}{3^{m+1}} - \frac{2}{3^{m+2}} \cdot \frac{1-\frac{1}{3^{\mu_m}}}{1-\frac{1}{3}}$$

$$\geq \frac{1}{3^{m+1}} - \frac{3^{\mu_m}-1}{3^{m+\mu_m+1}}$$

$$\geq \frac{1}{3^{m+\mu_m+1}}.$$

Therefore,

$$\frac{\Phi(x')-\Phi(x)}{x'-x} \geq \frac{\frac{1}{3^{m+\mu_m+1}}}{\frac{1}{3^{3m}}} \geq 3^{2m-\mu_m-1}.$$

Hence

$$\underset{x'=x}{Lim}\ \frac{\Phi(x')-\Phi(x)}{x'-x} > \underset{m=\infty}{Lim}\ 3^{2m-\mu_m-1} = \infty,$$

for $\underset{m=\infty}{Lim}\ (2m-\mu_m)=\infty$ by supposition.

Combining the results of (6.2), (6.3) and (6.4), we can assert that there exists an everywhere dense set of points at which $\Phi(x)$ has a right hand derivative equal to ∞.

(6.5) If $\sum\limits_{1}^{v-1}(a_{3r-1}+a_{3r})$ is odd, while the other conditions of (6.2) and (6.4) are satisfied by the representation of a point x, then it can be easily shown that there exists a right hand derivative at x, which has the value $-\infty$.

It follows, therefore, that there exists an everywhere dense set of points at each of which $\Phi(x)$ has a right hand derivative equal to $-\infty$.

(6.6) The preceding results can now be summarized as below:

(1) *There exists an everywhere dense set of points* S_1 *at each of which* $\Phi(x)$ *has a progressive derivative equal to* ∞. *A point* x *of this set is such that* (*a*) *in its representation*, $(a_{3n-1}+a_n)=4$ *for all values of* $n \geq v$, *and if* μ_m *denote the greatest number of* a_{3n+1}'*s*, $(n=m+1, m+2, \ldots)$ *immediately following* a_{3m+1} (*not equal to* 2), *which are equal to* 2, *then*

$$\underset{m=\infty}{Lim}\ (2m-\mu_m)=\infty\ ;$$

$$and \sum_{1}^{v-1} (a_{3r-1} + a_{3r}) \text{ is even.}$$

(2) *There exists an everywhere dense set of points S_2 at each of which $\Phi(x)$ has a progressive derivative equal to $-\infty$. A point x of this set is such that the condition (a) of (1) relating to the representation of x is satisfied, while*

$$\sum_{1}^{v-1} (a_{3r-1} + a_{3r}) \text{ is odd.}$$

Similarly it can be proved that:

(3) *There exists an everywhere dense set of points S_3 at each of which $\Phi(x)$ has a regressive derivative equal to ∞. A point x of this set is such that (a) in its representation $(a_{3n-1} + a_{3n}) = 0$ for all values of $n \geq v'$, and if μ'_m denote the greatest number of a_{3n+1}'s immediately succeeding a_{3m+1} (not equal to 0), which are all equal to 0, then*

$$\lim_{m=\infty} (2m - \mu'_m) = \infty;$$

$$and \sum_{1}^{v'-1} (a_{3r-1} + a_{3r}) \text{ is even.}$$

(4) *There exists an everywhere dense set of points S_4 at each of which $\Phi(x)$ has a regressive derivative equal to $-\infty$. A point x of this set is such that the condition (a) of (3) relating to its representation is satisfied, while $\sum_{1}^{v'-1} (a_{3r-1} + a_{3r})$ is odd.*

7. Second Example.

Let a point t in the interval $(0, 1)$ be represented as

$$t=\frac{a_1}{3}+\frac{a_2}{3.5}+\frac{a_3}{3^2.5}+\frac{a_4}{3^2.5^2}+\cdots,$$

where the a's are zero or positive integers such that $a_{2r} \leqq 4$ and $a_{2r+1} \leqq 2$ $(r=0, 1, 2,\ldots)$.

Corresponding to t let a number x be defined as

$$x=\varphi(t)=\frac{c_1}{3}+\frac{c_2}{3^2}+\frac{c_3}{3^3}+\cdots,$$

where

$$c_1=a_1,\ c_2=\mathrm{P}^{a_2}(a_3),\ \ldots,\ c_n=\mathrm{P}^{a_2+a_4+\cdots+a_{2n}}(a_{2n+1}),\ \ldots$$

and $\mathrm{P}^k(a)$ denotes a or $(2-a)$ according as k is even or odd.

I shall first prove that $\varphi(t)$ is a continuous function of t and then prove that it is non-differentiable.

(7.1) $x=\varphi(t)$ is a continuous function.

The numbers t may be divided into two classes:

(1) Those which are capable of double representation.

(2) Those which have a single representation only.

If t be a number of the second class, x is uniquely defined.

If t be a number of the first class, whose ending representation runs up to an odd number of terms, then, since,

$$\frac{a_1}{3}+\frac{a_2}{3.5}+\cdots+\frac{a_{2n+1}}{3^{n+1}.5^n}+\left(\frac{4}{3^{n+1}.5^{n+1}}+\frac{2}{3^{n+2}.5^{n+1}}\right)$$

$$+\left(\frac{4}{3^{n+2}.\,5^{n+2}}+\frac{2}{3^{n+3}.5^{n+2}}\right)+\ \ldots \text{ to infinity}$$

$$=\frac{a_1}{3}+\frac{a_2}{3.5}+\cdots+\frac{a_{2n+1}+1}{3^{n+1}.5^n},$$

t can be represented by the finite series or by the infinite series; and if $a_2+a_4+\cdots+a_{2n}$ is even, we see that the values obtained by applying the definition of x to the two modes of representation of t are

$$\frac{c_1}{3}+\frac{c_2}{3^2}+\cdots+\frac{c_n}{3^n}+\frac{c_{n+1}}{3^{n+1}}+\frac{2}{3^{n+2}}+\frac{2}{3^{n+3}}+\cdots$$

and

$$\frac{c_1}{3}+\frac{c_2}{3^2}+\cdots+\frac{c_n}{3^n}+\frac{c'_{n+1}}{3^{n+1}},$$

where $c_{n+1}=a_{2n+1}$, and $c'_{n+1}=a_{2n+1}+1$, so that the same value of x is obtained for both representations of t. Similarly if $a_2+a_4+\cdots+a_{2n}$ is odd, it can be seen that x has the same value for both representations of t. The case when the ending representation of t runs up to an even number of terms may be similarly treated.

$x=\varphi(t)$ is thus a single-valued function of t. It is continuous, for if t and t' are identical as regards their first $2n$ terms, the corresponding x and x' are identical as regards their first n terms, and, therefore, when t' tends to t with increasing n, x' tends to x.

(7.2) $x = \varphi(t)$ **is a non-differentiable function.**

For proving the non-differentiability of $\varphi(t)$, the numbers t may be divided into two classes (A) and (B).

First consider class (A), i.e., the class of numbers in which $a_{2r} \leqq 2$ for infinitely many values of r.

If t_1 be a point of this class, we see that the addition of $\frac{2}{3^r.5^r}$ to t_1 does not make any change in the value of $\varphi(t_1)$, since a_{2r} becomes $a_{2r}+2$ so that k remains even or odd as before; while the addition of $\frac{1}{3^{r+1}.5^r}$ to t_1 does so. Therefore,

$$\lim \left| \frac{\varphi\left(t_1 + \frac{2}{3^r.5^r}\right) - \varphi(t_1)}{\frac{2}{3^r.5^r}} \right| = 0$$

and

$$\overline{\lim} \left| \frac{\varphi\left(t_1 + \frac{1}{3^{r+1}.5^r}\right) - \varphi(t_1)}{\frac{1}{3^{r+1}.5^r}} \right| = \infty$$

where the limits are taken as r tends to infinity assuming those values for which the inequality $a_{2r} \leqq 2$ is satisfied.

Thus at the points of class (A) the differential coefficient is non-existent.

Now consider class (B), i.e., those numbers t in which $a_{2r} > 2$ for infinitely many values of r.

If t_2 be a point of this class, we see that the subtraction of $\dfrac{2}{3^r.5^r}$ from t_2 does not change the value of $\varphi(t_2)$, while the subtraction of $\dfrac{1}{3^{r+1}.5^r}$ from t_2 does so, so that

$$\lim \left| \frac{\varphi\left(t_2 - \dfrac{2}{3^r.5^r}\right) - \varphi(t_2)}{\dfrac{2}{3^r.5^r}} \right| = 0$$

and

$$\overline{\lim} \left| \frac{\varphi\left(t_2 - \dfrac{1}{3^{r+1}.5^r}\right) - \varphi(t_2)}{\dfrac{1}{3^{r+1}.5^r}} \right| = \infty,$$

where the limits are taken as r tends to infinity assuming those values for which the inequality $a_{2r} > 2$ is satisfied.

Thus at the points of class (B) the differential coefficient is non-existent. $x = \varphi(t)$ is, therefore, a non-differentiable function in the interval (0, 1).

Every function of this type* can, by a similar treatment, be shown to be devoid of a differential coefficient at each point in (0, 1).

*For the general class of functions of which the above is a particular case see Singh (79).

8. Third Example. Let the numbers of the interval (0, 1) be expressed in the decimal scale as

$$x = \frac{a_1}{10} + \frac{a_2}{10^2} + \cdots + \frac{a_{2n-1}}{10^{2n-1}} + \frac{a_{2n}}{10^{2n}} + \cdots$$

where every a is one of the numbers 0, 1, 2,...or 9. Corresponding to x we define a number

$$y = f(x) = \frac{b_1}{20} + \frac{b_2}{20^2} + \frac{b_3}{20^3} + \cdots + \frac{b_n}{20^n} + \cdots$$

where $b_n = \pm (2a_{2n-1} + c_n)$, and $c_n = 0$, 1 or 2 according as a_{2n} is 0, 2, 4 or 1, 3, 5, 7, 9 or 6, 8 respectively, and b_n‡ has the same sign as b_{n-1}, if a_{2n-2} is 0, 2, 4, 5, 7 or 9, otherwise it has the opposite sign; and b_1 is always positive.

(8.1) $y=f(x)$ **is a continuous function.** y is evidently continuous at all the points x in (0, 1) which have a unique non-terminating representation in the scale of ten. Those points x which have a terminating representation have also a non-terminating representation. To prove the continuity of y in (0, 1) it will be enough to show that y is uniquely determined at all the points where x has a double representation.

Let x be a point whose representation runs up to $(2n-1)$ places. Then

$$\begin{aligned} x &= \frac{a_1}{10} + \frac{a_2}{10^2} + \cdots + \frac{a_{2n-1}}{10^{2n-1}} \\ &= \frac{a_1}{10} + \frac{a_2}{10^2} + \cdots + \frac{(a_{2n-1} - 1)}{10^{2n-1}} + \frac{9}{10^{2n}} + \frac{9}{10^{2n+1}} + \cdots \end{aligned}$$

‡ It will be observed that b_n can have the value 20.

And according to our definition, corresponding to the two representations of x

$$y = \frac{b_1}{20} + \frac{b_2}{20^2} + \cdots + \frac{b_{n-1}}{20^{n-1}} + \frac{b_n}{20^n}$$

and

$$= \frac{b_1}{20} + \frac{b_2}{20^2} + \cdots + \frac{b_{n-1}}{20^{n-1}} + \frac{b'_n}{20^n}$$
$$\pm\left(\frac{19}{20^{n+1}} + \frac{19}{20^{n+2}} + \cdots\right).$$

We have $b_n = \pm\ (2a_{2n-1}+0)$ and $b'_n = \pm\ [2(a_{2n-1}-1)+1]$, i.e., $|\ b'_n\ | = |\ b_n\ | -1.$ But the terms that follow b'_n have the same sign as b'_n and their sum is $\frac{1}{20^n}$. Hence the same value of y corresponds to the two representations of x.

Similarly, if x has a terminating representation running up to $2n$ places, we can show that the same value of y is obtained for both representations of x.

It follows that y is a continuous function of x in $(0, 1)$.

(8.2) y **is a non-differentiable function.** (a) Let x be a point in whose representation an infinite number of a_{2n}'s are 0, 1 or 2. Then

$$f\left(x + \frac{2}{10^{2n}}\right) - f(x) = 0$$

while

$$\left| f\left(x + \frac{5}{10^{2n}}\right) - f(x) \right| = \frac{1}{20^n},$$

for an infinite number of values of n tending to infinity, so that one of the derivates at x is zero, whilst another is indefinitely great* (numerically). Thus the differential coefficient is non-existent at all such points.

(b) Let x be a point in whose representation an infinite number of a_{2n}'s are 3 or 4. Then

$$f\left(x - \frac{2}{10^{2n}}\right) - f(x) = 0$$

while

$$\left| f\left(x + \frac{5}{10^{2n}}\right) - f(x) \right| = \frac{1}{20^n},$$

and, therefore as before, the differential coefficient is non-existent at all such points.

(c) Let x be a point in whose representation an infinite number of a_{2n}'s are 5, 6 or 7. Then

$$f\left(x + \frac{2}{10^{2n}}\right) - f(x) = 0$$

while

$$\left| f\left(x - \frac{5}{10^{2n}}\right) - f(x) \right| = \frac{1}{20^n}.$$

*For $\displaystyle \operatorname*{Lim}_{h=0} \left| \frac{f(x+h) - f(x)}{h} \right| \geqq \operatorname*{Lim}_{n=\infty} \frac{\frac{1}{20^n}}{\frac{5}{10^{2n}}}$

and, therefore as before, the differential coefficient is non-existent at all such points.

(d) Let x be a point in whose representation an infinite number of a_{2n}'s are 8 or 9. Then

$$f\left(x-\frac{2}{10^{2n}}\right)-f(x)=0$$

while

$$\left|f\left(x-\frac{5}{10^{2n}}\right)-f(x)\right|=\frac{1}{20^n},$$

and, therefore as before, the differential coefficient is non-existent at all such points.

Now, any point in (0, 1) comes under at least one of the four heads enumerated above, and hence at no point x in (0, 1) does there exist a differential coefficient.*

* For the general class of functions of which the above is a particular case see Singh (83).

FOURTH LECTURE

PROPERTIES OF NON-DIFFERENTIABLE FUNCTIONS

1. In to-day's lecture I shall give an account of the recent work relating to the study of the properties of non-differentiable functions, especially with regard to the existence of cusps and derivatives. In this connection I shall also enumerate some of the important results of the theory of derivates which have a direct bearing on our subject. An account will also be given of the character of the oscillations of non-differentiable functions. It will be shown by means of an example that the oscillations of a non-differentiable function may be *unenumerable* in every interval how-so-ever small taken in the domain of the function. Whether or not this is a general characteristic of all non-differentiable functions is unknown.

2. **Upper and lower derivates.** When a continuous function $f(x)$ does not possess a right (left) derivative, the incrementary ratio

$$R(x, h) = \frac{f(x+h) - f(x)}{h},$$

$h > 0$ (< 0), does not tend to any limit as h tends to zero. In such a case $R(x, h)$ is associated with its upper and lower limits which are called the upper and lower derivates of $f(x)$ on the right (left), and are denoted respectively by the symbols $D^+f(x)$, $D_+f(x)$ $[D^-f(x), D_-f(x)]$. In recent years, these derivates have been closely studied and a number of very interesting results obtained. Whenever the derivates are finite *almost everywhere*, they can be used in the place of the differential coefficient. The derivates of non-differentiable functions, however, can not be used as they are infinite *almost everywhere*. In fact, in the case of every non-differentiable function, it can be shown that

$$(i)\ \mathrm{D}^+ = \mathrm{D}^- = \infty$$

and

$$(ii)\ \mathrm{D}_+ = \mathrm{D}_- = \infty$$

almost everywhere.

3. **Sufficient condition for differentiability.** The discovery of continuous non-differentiable functions brought to the fore-front the question: "In what cases and for what aggregates of values of x can we assert that a function $f(x)$ possesses a differential coefficient?" Perhaps the most important answer to this question has been given by the following theorem of Lebesgue ([52]):

A continuous function of bounded variation has a finite differential coefficient at every point which does not belong to a set of measure zero.

This theorem was extended to the case of monotone functions (not necessarily continuous) by W. H. Young and G. C. Young ([104]), who gave a proof independent of the notion of integration and of transfinite numbers. Elementary demonstrations of the theorem have also been given by Faber ([26]) and Tonelli ([93]). Proofs which hold for any function of bounded variation have been given by Carathéodory ([16]) Steinhaus ([88]) and Rajchman and Saks ([68]). It has been shown by Singh ([87]) that the set of points of differentiability of a function of bounded variation is of the second category.

It follows from Lebesgue's theorem that a non-differentiable function is *not a function of bounded variation* and is consequently not *rectifiable.* This leads us to the following three properties of a non-differentiable function:

(i) it is everywhere oscillating;
(ii) the length of the arc between any two points on the curve is infinite; and
(iii) the geometrical graph of the function can not be drawn.

4. **Existence of cusps.** The question arises whether a non-differentiable function can possess proper maxima and minima. As the function is non-differentiable, the proper maxima and minima, if they exist, must be either cusps or edge-points (*i.e.*, points at which the derivates on one side are positive whilst those on the

other side are negative; the derivates on one side at least being finite). Hardy (32) and G. C. Young (100) have shown that Weierstrass's function

$$W(x)=\sum_{o}^{\infty} a^n \, Cos\ (b^n \pi x)$$

has at the everywhere dense set of points $x=\frac{2r}{b^m}$ right-hand derivatives $=-\infty$, and left-hand derivatives $=\infty$. Thus these points are cusps whose edges point upwards and are proper maxima of Weierstrass's function. It has also been shown that at the set of points $x=\frac{2r+1}{b^m}$, the function has proper minima, with cusps pointing downwards. Bhar (8) has found similar everywhere dense set of cusps on the curve

$$D(x) = \sum_{1}^{\infty} a^n \, Sin\ (b^n \pi x).$$

All the functions defined by series and given in the first lecture, can be shown to possess cusps at everywhere dense sets of points. Hobson* has remarked, "It does not appear to be definitely known whether a non-differentiable function can exist which has no cusps." This question was answered by me in the third lecture, where it was shown that the function $\varphi_{3,1,3}(t)$ has no cusps, because at no point does the function possess

* See Hobson (36) p. 405.

both a progressive and a regressive derivative. Functions which do not possess cusps were considered by Moore ([57]) Sierpinski ([76]) and Hahn ([31]), but their investigations seem to have escaped notice.

It does not appear to be known whether a non-differentiable function can exist which has edge-points at an everywhere dense set.

(4.1) Theorems about cusps and derivates. B. Levi ([54]) has proved the theorem :

The aggregate of points x, of an interval (a, b), at which a function $F(x)$, continuous in (a, b), possesses a progressive and a regressive derivative, which are different from each other, is enumerable.

This theorem has been generalised by Rosenthal ([69]) and G. C. Young ([98]).

From Levi's theorem it follows that the set of points at which a function has cusps is enumerable. In connection with the existence of an everywhere dense set of cusps may also be mentioned the following theorems due to Koenig ([48]) and Rosenthal ([69]) respectively.

(1) *If the continuous function $f(x)$ possess cusps at an everywhere dense set of points, there exists an everywhere dense set of points at each of which the differential coefficient has the prescribed value c, or is*

indeterminate and such that c lies between its upper and lower limits.

(2) *If the continuous function $f(x)$ has cusps at an everywhere dense set of points, three exists a set of the second category at each point of which the function lacks a progressive and a regressive derivative.*

(4.2) **Denjoy's Theorem.** The relations which subsist between the four derivates of a continuous function at a point, if one disregards sets of measure zero, have been systematically investigated by Denjoy(21). He has obtained the following theorem :

*If $f(x)$ be a continuous function, finite at each point, and if a set of measure zero be left out of account, then, at the various points x only the following four cases are possible**.

(1) $D^+ = D^- = D_+ = D_- =$ finite,

(2) $D^+ = D^- = \infty$; $D_+ = D_- = -\infty$,

(3) $D^+ = \infty$, $D_- = -\infty$; $D_+ = D^- =$ finite,

(4) $D^- = \infty$, $D_+ = -\infty$; $D_- = D^+ =$ finite.

Each of the above four cases can be individually realized, *i.e.*, a function can be constructed for which a definite one of the four cases occurs. Denjoy has

* The above result has been shown to hold for a finite and measurable function by G. C. Young (101). Saks (71) has given a proof applicable to non-measurable functions, and an extension has been made by G. C. Young to the case in which the function is infinite at the points of a set of positive measure.

constructed a function for which every one of these four cases occurs.

(4.3) **Derivates of non-differentiable functions.** The properties of the derivates of non-differentiable functions have been studied by W. H. Young ([103]), and the following result has been established by him:

If a function $f(x)$ is non-differentiable in an interval, then

(1) there is necessarily a distinction of right and left in the values of the derivates at a set of points which is everywhere dense and is of the first category ;

(2) the upper and lower bounds of the values of the derivates at the points of this set are respectively ∞ and $-\infty$;

(3) at the remaining points of the interval, both the upper derivates are ∞ and both the lower derivates are $-\infty$, exception being made of at most another set of the first category.

Young's theorem is supplemented by the following result due to Singh ([81]) :

Every finite and continuous non-differentiable function has associated with it

(1) an everywhere dense set of points S_1, at which $D_+ > M$,

(2) an everywhere dense set of points S_2, *at which* $D^+ < -M$,

(3) an everywhere dense set of points S_3, *at which* $D_- > M$,

(4) an everywhere dense set of points S_4, *at which* $D^- < -M$,

where M is any positive number, however large.

Although the above result does not establish the existence of points at which infinite derivatives exist, it nevertheless shows that, in the case of every continuous non-differentiable function, there exist everywhere dense sets of points at each point of which there is an arbitrarily near approximation to the existence of a determinate (right and left) derivative which is numerically infinite.

5. Study of Derivates. G. C. Young ([100]) has made a detailed study of the derivates of Weierstrass's function

$$W(x) = \sum a^n \operatorname{Cos} (b^n \pi x),$$

where b is an odd integer and $ab > 1 + \frac{3\pi}{2}$.

The result obtained by her may be summarised as follows:

Let a point x in $(0, 1)$ be expressed in the scale of b, as

$$x = \frac{c_1}{b} + \frac{c_2}{b^2} + \cdots + \frac{c_n}{b^n} + \cdots\cdots$$

where the c's are 0, 1, 2,.., $b-1$, then,

(1) at the set of points S_1 represented by an ending series of the above form $W(x)$ has cusps ;

(2) at the set of points S_2 given by the points x in whose representation $c_{m+1}, c_{m+2}, \ldots\ldots$ are all even, the derivates are non-symmetrical ;

(3) at the set of points S_3 which is complementary to (S_1+S_2), the derivates are symmetrical, both the upper derivates being ∞ and both the lower derivates $-\infty$.

The set S_1 is enumerable and the set S_2 can be easily proved to be of zero measure. W. H. Young's theorem is thus verified. A closer study of the derivates of Weierstrass's function seems to be necessary in order to find the actual values of the derivates at the various points of the set S_2.

The investigations of Porter ([63]) and Bhar ([8]), although incomplete, show that the derivates of Dini's function

$$\sum_{1}^{\infty} a^n \, Sin\, (b^n \pi x),$$

where b is an even integer, and $ab > 1 + \frac{3\pi}{2}$, have the same general character as those of Weierstrass's function. Let the numbers x in $(0, 1)$ be expressed as radix fractions in the scale of b as

$$x = \frac{c_1}{b} + \frac{c_2}{b^2} + \cdots + \frac{c_n}{b^n} + \cdots\cdots \quad .$$

According to Porter, except at the points of the null set $[x]$ for which $c_n = o$ and $c_n = b-1$ fail to occur infinitely often, the right and left incremental ratios of Dini's function have each for their upper and lower limits $+\infty$ and $-\infty$.

Bhar has studied the function

$$D(x) = \sum \frac{Sin\ (16^n \pi x)}{2^n}.$$

His method can be modified to prove that the function

$$\sum a^n\, Sin\ (b^n \pi x)$$

possesses cusps at an everywhere dense set of points in $(0, 1)$.

6. The Derivates of $\phi_{m,r,p}(x)$. In the third lecture I showed how the set of points where the function $\phi_{3,1,3}(x)$ possesses one-sided differential co-efficients may be obtained. I shall now state the similar result which can be obtained for the general class of functions $\phi_{m,r,p}(x)$.

$\phi_{m,r,p}(x)$ is nowhere differentiable in $(0, 1)$, has no cusps, and

(1) there exists an everywhere dense set S_1, at which $\phi_{m,r,p}(x)$ has a right hand derivative equal to ∞.

A point x of this set is such that (a) in its representation

$$a_{m(n-1)+r+1}+a_{m(n-1)+r+2}+\cdots+a_{mn+r-1}=(m-1)(p-1)$$

for all values of $n \geq \nu$ (a fixed number), and if μ_q denote the greatest number of a_{mn+r}'s $(n=q+1,\ q+2,\ \ldots\ldots)$ immediately succeeding a_{mq+r} (not equal to $p-1$) which are all equal to $p-1$, then

$$\lim_{q\to\infty} (m-1)q-\mu_q=\infty,$$

and (*b*) $\sum\limits_{n=o}^{\nu-1}(a_{m(n-1)+r+1}+\cdots+a_{mn+r-1})$ *is even;*

(2) *There exists an everywhere dense set S_2 at which $\phi_{m,r,p}(x)$ has a right hand derivative equal to $-\infty$. A point x of this set is such that the condition (a) of (1) is satisfied while,*

$$\sum_{n=o}^{\nu-1}(a_{m(n-1)+r+1}+\cdots+a_{mn+r-1}) \text{ is odd;}$$

(3) *There exists an everywhere dense set S_3 at which $\phi_{m,r,p}(x)$ has a left hand derivative equal to ∞. A point x of this set is such that (a) in its representation $(a_{m(n-1)+r+1}+a_{m(n-1)+r+2}+\cdots\cdots+a_{mn+r-1})=o$ for all values of $n\geq \nu$, and if μ'_q denote the greatest number of a_{mn+r}'s $(n=q+1,\ q+2,\ldots)$ immediately succeeding a_{mq+r} (not equal to 0) which are all equal to 0, then*

$$\lim_{q\to\infty} (m-1)q-\mu_q'=\infty$$

and (*b*) $\sum_{n=o}^{\nu-1} (a_{m(n-1)+r+1} + \cdots + a_{mn+r-1})$ *is even ;*

(4) *there exists an everywhere dense set* S_4 *at which* $\Phi_{m,r,p}(x)$ *has a left hand derivative equal to* $-\infty$. *A point* x *of this set is such that the condition* (*a*) *of* (*3*) *is satisfied, while*

$$\sum_{n=o}^{\nu-1} (a_{m(n-1)+r+1} + \ldots + a_{mn+r-1}) \text{ is odd.}$$

7. **The Oscillations of non-differentiable functions.** Attempts were made by Wiener [97], Klein [40] and G. C. Young [100] to evolve a "graphical representation" of Weierstrass's function. As the function does not possess a graph, exact information as to the nature of the singularity at a point on the curve is obtained mainly by the study of the values of the derivates at that point. For instance, if we know that at a point P (x_1, y_1) on a continuous curve $y = f(x)$, $D^+ = \infty$ and $D_+ = -\infty$, we can at once say that the curve cuts the line $y = y_1$ at an indefinitely large number of points in any neighbourhood on the right of P, and so makes an infinite number of oscillations. Similar will be the case on the left of P, if $D^- = \infty$ and $D_- = -\infty$. It has been shown by G. C. Young [100] that for a set of points $[x]$ whose measure is 1, in (0, 1), both the upper derivates of Weierstrass's function are ∞ and both the

lower derivates are $-\infty$. It follows, therefore, that a point P on the curve which corresponds to a point x belonging to the set $[x]$ has the property that any line through P cuts the curve an infinite number of times on both sides of P.

(7.1) It is evident that Weierstrass's function

$$W(x)=\sum a^n \, Cos\,(b^n \pi x)$$

(where b is an odd interger) is zero at $x=\frac{1}{2}$. At this point, the derivates on the right as well as on the left oscillate between ∞ and $-\infty$. It follows that the point $x=\frac{1}{2}$ is a limiting point of the zeros of the function $W(x)$. A set of zeros of $W(x)$ with $x=\frac{1}{2}$ as a limiting point has been actually located by G. Prasad(67).

His result in the case of the function

$$W(x)=\sum \frac{Cos\,(13^n \pi x)}{2^n}$$

may be stated as follows:

There is a zero of $W(x)$ between

(*i*) $\left(\frac{1}{2}\pm\frac{1}{13^k}\right)$ and $\left(\frac{1}{2}\pm\frac{3/2}{13^k}\right)$;

(*ii*) another between

$$\left(\frac{1}{2}\pm\frac{3/2}{13^k}\right) \text{ and } \left(\frac{1}{2}\pm\frac{2}{13^k}\right);$$

(*iii*) a third between

$$\left(\frac{1}{2}\pm\frac{3}{13^k}\right) \text{ and } \left(\frac{1}{2}\pm\frac{7/2}{13^k}\right);$$

* See also Sharma (75). For the zeros of Dini's function see Mookerji (56).

(*iv*) a fourth between

$$\left(\frac{1}{2}\pm\frac{7/2}{13^k}\right) \text{ and } \left(\frac{1}{2}\pm\frac{4}{13^k}\right);$$

(*v*) a fifth between

$$\left(\frac{1}{2}\pm\frac{5}{13^k}\right) \text{ and } \left(\frac{1}{2}\pm\frac{11/2}{13^k}\right);$$

(*vi*) and a sixth between

$$\left(\frac{1}{2}\pm\frac{11/2}{13^k}\right) \text{ and } \left(\frac{1}{2}\pm\frac{6}{13^k}\right).$$

The above is not a complete list of all the zeros of the function $W(x)$. The determination of all the zeros is probably not practically possible. Bhar ([9]) has given a list of some of the zeros of the function

$$\sum \frac{16^n \begin{matrix} Cos \\ Sin \end{matrix} \left\{ 1.\,3.\,5.\,\ldots\,(2n-1) \right\} \pi x}{1.\,3.\,5.\,\ldots\,(2n-2)}.$$

Prasad's result quoted above is a verification of the conclusion that we can arrive at by a consideration of the values of the derivates at the point $x=\frac{1}{2}$. It would be interesting to find some special character of the set of zeros, *e.g.*, whether they form an enumerable set or not, or whether the set is closed or open.

(7.2.) Very interesting results regarding the nature of the oscillations of a non-differentiable function have been recently obtained* by studying the intersections of the line $y=c$ with the curve $y=\Phi_{m,r,p}(x)$ given in the third lecture. It has been found that the roots of

*See Singh ([85]). For a similar study of another function see Singh ([84]).

the equation $\Phi_{m,r,p}(x) = c$ $(0 \leq c \leq 1)$, form a set S_c which is perfect and is of zero measure. Thus the oscillations of the function $\Phi_{m,r,p}(x)$ in every interval, ever-so-small, are unenumerable. The function is thus much more complicated than ordinary transcendental functions which may have an enumerably infinite number of oscillations in an interval. It may be pointed out that by the help of such a function we can easily express the continuum in (0, 1) as an unenumerable aggregate of unenumerable aggregates.

8. The Set S_c of the roots of $\Phi_{3,1,3}(x) = c$. I shall now find the roots of the equation $\Phi_{3,1,3}(x) = c$ and prove that they form a perfect set S_c of measure zero in the interval (0, 1). The proof for the general case can be similary carried out. Taking the definition of $\Phi_{m,r,p}(x)$ given in the third lecture, we have, for values of x, in (0, 1), expressed in the scale of 3 as

$$x = \frac{a_1}{3} + \frac{a_2}{3^2} + \frac{a_3}{3^3} + \cdots + \frac{a_{3n-1}}{3^{3n-1}} + \frac{a_{3n}}{3^{3n}} + \frac{a_{3n+1}}{3^{3n+1}} + \cdots,$$

$$\Phi_{3,1,3}(x) = \frac{b_1}{3} + \frac{b_2}{3^2} + \cdots + \frac{b_n}{3^n} + \cdots,$$

where

$$b_1 = k^o(a_1) = a_1,$$

$$b_2 = k^{a_2 + a_3}(a_4),$$

$$b_3 = k^{a_2+a_3+a_5+a_6}(a_7),$$

....

$$b_n = k^{a_2+a_3+a_5+a_6+\cdots+a_{3(n-1)-1}+a_{3(n-1)}}(a_{3(n-1)+1}),$$

and so on; and where $k^s(a) = a$ or $(2-a)$ according as s is even or odd.

Let the constant c have the following representation when expressed in the scale of 3:

$$c = \frac{c_1}{3} + \frac{c_2}{3} + \frac{c_3}{3^3} + \cdots + \frac{c_n}{3^n} + \cdots\cdots$$

Then the a's in the representation of

$$x_c = \frac{a_1}{3} + \frac{a_2}{3^2} + \frac{a_3}{3^3} + \cdots + \frac{a_{3n-1}}{3^{3n-1}} + \frac{a_{3n}}{3^{3n}} + \frac{a_{3n+1}}{3^{3n+1}} + \cdots$$

which corresponds to the c_n's must satisfy the following conditions:

$$a_1 = c_1;$$

and if $c_{n+1} = 0$, then $a_{3n+1} = 0$ or 2 according as

$$\sum_{r=1}^{n} (a_{3r-1} + a_{3r}) \text{ is even or odd;}$$

if $c_{n+1} = 2$, then $a_{3n+1} = 2$ or 0 according as

$$\sum_{r=1}^{n} (a_{3r-1} + a_{3r}) \text{ is even or odd;}$$

and if $c_{n+1} = 1$, then $a_{3n+1} = 1$ whatever

$$\sum_{r=1}^{n} (a_{3r-1} + a_{3r}) \text{ may be.}$$

It is obvious that there are an infinite number of x_c's, the a's in whose representation satisfy the conditions given above. The points x_c form a set S_c, which is the set of the roots of $\Phi_{3,1,3}(x)=c$ and which we now propose to study.

(8.1) *S_c is unenumerable.* This follows from the fact that S_c is perfect.

(8.2) *S_c is perfect.* Let

$$x_c = \frac{a_1}{3}+\frac{a_2}{3^2}+\frac{a_3}{3^3}+\cdots+\frac{a_{3n-1}}{3^{3n-1}}+\frac{a_{3n}}{3^{3n}}+\frac{a_{3n+1}}{3^{3n+1}}+\cdots$$

be a root of $\Phi_{3,1,3}(x)=c$.

Then the point

$$x'_c=\frac{a_1}{3}+\frac{a_2}{3^2}+\cdots\cdots+\frac{a'_{3n-1}}{3^{3n-1}}+\frac{a'_{3n}}{3^{3n}}+\frac{a_{3n+1}}{3^{3n+1}}+\cdots\cdots,$$

which differs from x_c at the $(3n-1)$th and $3n$th places only is also a root of $\Phi_{3,1,3}(x)=c$, if

$$\left|\,(a_{3n-1}+a_{3n})-(a'_{3n-1}+a'_{3n})\,\right| =2 \text{ or } 4.$$

By letting n tend to infinity, we see that points x'_c belonging to S_c can be found as near to x_c as we please. Thus x_c is a limiting point of the points of S_c. *It follows that the set S_c is dense-in-itself.*

That the set S_c is closed follows from the continuity of $\Phi_{3,1,3}(x)$.

Therefore, the set S_c is perfect.

(8.3) *S_c has zero measure.* In the representation of c, if $c_1=0$, then there are 2 intervals

$$[\tfrac{1}{3}, \tfrac{2}{3}] \text{ and } [\tfrac{2}{3}, 1]$$

in each of which there is no point of S_c; if $c_1 = 1$, then there are two intervals

$$[0, \tfrac{1}{3}] \text{ and } [\tfrac{2}{3}, 1]$$

in each of which there is no point of S_c; and similarly if $c_1 = 2$, we find that there are two intervals

$$[0, \tfrac{1}{3}] \text{ and } [\tfrac{1}{3}, \tfrac{2}{3}]$$

in each of which there is no point of S_c.

We thus find that, *whatever c_1 may be*, there are two intervals that do not contain points of S_c, and that the sum of their lengths is $\frac{2}{3}$.

Again, *whatever c_1 and c_2 may be*, there are besides the two intervals of the above type, 2.3^2 more intervals, each of length $\frac{1}{3^4}$ which do not contain points of S_c. For supposing $c_2 = 2$, a_2 and a_3 must respectively have the values

$$1, 0 \text{ or } 0, 1 \text{ or } 1, 2 \text{ or } 2, 1,$$

whilst $a_4 = 0$, otherwise a_2 and a_3 are respectively

$$0, 0 \text{ or } 0, 2 \text{ or } 2, 0 \text{ or } 1, 1 \text{ or } 2, 2,$$

whilst $a_4 = 2$. Thus (for the case $c_1 = 1$, $c_2 = 2$) there can not be points of the set S_c in the two intervals

$$\left\{ \frac{1}{3} + \frac{1}{3^2} + \frac{0}{3^3} + \frac{1}{3^4}, \frac{1}{3} + \frac{1}{3^2} + \frac{0}{3^3} + \frac{2}{3^4} \right\}$$

and

$$\left\{ \frac{1}{3} + \frac{1}{3^2} + \frac{0}{3^3} + \frac{2}{3^4}, \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} \right\},$$

or again in the two intervals

$$\left\{ \frac{1}{3} + \frac{0}{3^2} + \frac{1}{3^3} + \frac{1}{3^4},\ \ \frac{1}{3} + \frac{0}{3^2} + \frac{1}{3^3} + \frac{2}{3^4} \right\}$$

and

$$\left\{ \frac{1}{3} + \frac{0}{3^2} + \frac{1}{3^3} + \frac{2}{3^4},\ \ \frac{1}{3} + \frac{0}{3^2} + \frac{2}{3^3} \right\},$$

and so on for all the nine cases, thus making a total of 2.3^2 intervals that do not contain points of S_c. It follows that *whatever c_1 and c_2 may be*, there are a set of intervals whose total length is

$$\frac{2}{3} + \frac{2}{3^2},$$

which do not contain points of S_c.

Similarly, it can be shown that *whatever c_1, c_2 and c_3 may be*, there are a set of intervals which do not contain points of S_c, and that the sum of the lengths of these intervals is

$$\frac{2}{3} + \frac{2}{3^2} + \frac{2}{3^3}.$$

Proceeding in this manner, we find that *whatever the c's may be*, there are a set of intervals which do not contain points of S_c, and whose measure is

$$\frac{2}{3} + \frac{2}{3^2} + \frac{2}{3^3} + \cdots + \frac{2}{3^n} + \cdots\cdots = 1.$$

Therefore, S_c has zero measure.

9. **The Mean Differential Coefficient.** If $f(x)$ be continuous at the point x, the mean differential coefficient at x is the limit, if it exists, of the ratio

$$\frac{f(x+h)-f(x-h)}{2h},$$

as h tends to zero.

G. Prasad [65] has shown that Weierstrass's function possesses finite mean differential coefficients at an everywhere dense set of points. The functions defined by Darboux [20], Lerch [53], Faber [25, 26, 27], Landsberg [51], Steinitz [89], Singh [78, 79, 83, 85] and Hahn [31] possess mean differential coefficients. It has been stated by Bhar [7] that Dini's function $\sum a^n Sin\ (b^n \pi x)$ does not possess a mean differential coefficient.*

10. **Remarks**. We have traced the gradual development of our knowledge regarding the nature and properties of non-differentiable functions, and have discussed some of the problems that have arisen during the course of the study of such functions. We have also pointed out some of the advances that have been made in other branches of the theory of functions due to that study. But the fundamental question which the discovery of non-differentiable functions has raised remains still unanswered. It is this: 'What minimum restrictions should be placed on a function so that it will possess a differential coefficient at each point of its domain of definition?' Or, in other words, 'Does there

* Bhar (9), p. 80, points out a slip in his paper (7), which will require a recasting of his proof for the non-existence of the mean differential coefficient.

exist a necessary and sufficient condition for the differentiability of a function in an interval ?' The question has been engaging the attention of mathematicians since the publication of Weierstrass's example of a non-differentiable function in 1874, and yet we are no nearer its solution.

It is well-known that continuity is *necessary* for differentiability, but it is not *sufficient* as is shown by the existence of continuous non-differentiable functions. The restriction of *bounded variation* has also proved insufficient. Although a continuous function of bounded variation must possess a differential coefficient *almost everywhere*, yet there are examples of such functions which do not possess differential coefficients at unenumerable everywhere dense sets of points. The same remark applies to absolutely continuous functions. The discovery of a necessary and sufficient condition for differentiability will no doubt be a great advance and will, I believe, find immediate application in geometry and in physics. But with the present state of our knowledge it does not appear to be possible to discover any such condition. Further study of the theory of aggregates and perhaps a closer classification of functions may give us the key to the solution.

It has been suggested by W. H. Young and G. C. Young that efforts should be made to evolve a definition of differentiation, according to which all continuous functions would be differentiable, at least *almost everywhere*, and correspondingly to find a definition of integration that would always lead back from the derivate to the primitive function. We have seen that in the case of non-differentiable functions the upper and lower right (left) derivates have the values $+\infty$ and $-\infty$ respectively *almost everywhere*. At such points, therefore, by choosing a suitable method of approach the incrementary ratio can be made to converge to any desired value between $+\infty$ and $-\infty$. The problem, then, is to devise a method of choosing h's which would not only provide finite derivatives *almost everywhere*, but would also suggest a corresponding integrating process, leading from such a derivative back to the primitive function.

A suggestion has been made by G. C. Young to use (what she calls) the *mean symmetric derivate*, which she has defined and shown to exist in the case of a continuous function, except at an enumerable set of points; but there is no evidence as yet to show that it has any practical value. The *mean symmetric derivate* of Weierstrass's function is 1 *almost everywhere*, and is clearly of no particular use. Likewise the notion of the generalised Riemann derivative of the nth order does not appear to take us appreciably nearer to the solution of the problem.

The successful generalisations of the notion of integration and that of summation have shown that it is possible to select and utilize one out of an infinite number of limits. In fact, if by the application of the theory of sequences, or otherwise, we can devise some method of selection, the plurality of limits can be turned to an advantage. For the solution of our problem, however, we cannot hope much from the theory of sequences alone. The success of the method of sequences is intrinsically due to the mechanism of monotony, but monotony in such a connection as this can serve no useful purpose. It is hoped that a more profound study of the theory of aggregates may lead to the solution of the problem, or at least to the determination of the limits between which the solution should lie.

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