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AN EFFICIENT CONSTRUCTION OF REAL NUMBERS

Ross Street

The ordered field $\mathbf{R}$ of real numbers can be constructed directly from the ring $\mathbf{Z}$ of integers without first manufacturing the field $\mathbf{Q}$ of rationals and without (explicitly) using Cauchy sequences or Dedekind cuts. This idea has avoided the wide publicity it deserves. I believed the original discoverer to be Steve Schanuel (SUNY, Buffalo, New York) who explained it to me while he was visiting Macquarie University last year. Peter Johnstone has recently told me that the construction was also proposed by Richard Lewis (Sussex, England). Does anyone know of other independent discoverers?

By way of motivation notice that a real number α determines a function $f : \mathbf{Z} \to \mathbf{Z}$ given by $f(n) = [\alpha n]$, where square brackets denote "integer part". Then $f(10^r)/10^r$ approximates α to r decimal places. In fact $f(n)/n \to \alpha$ as $n \to \infty$. If α is an integer then f preserves addition; otherwise it almost does in that $|f(m+n) - f(m) - f(n)| \leqslant 3$.

Now to the construction. A function $u : X \to \mathbf{Z}$ is called *bounded* when its image is finite; that is, when there exists $k \in \mathbf{Z}$ such that $|u(x)| \leqslant k$ for all $x \in X$. The set $\mathbf{Z}^X$ of functions from X to $\mathbf{Z}$ is an abelian group under pointwise addition $(f + g)(x) = f(x) + g(x)$. For X an abelian group, define $f \in \mathbf{Z}^X$ to be a *quasi-homomorphism* when $f(x + y) - f(x) - f(y)$ is bounded as a function of $(x,y) \in X \times X$. The quasi-homomorphisms form a subgroup $qh(X,\mathbf{Z})$ of $\mathbf{Z}^X$, and, the bounded functions form a subgroup of $qh(X,\mathbf{Z})$.

The abelian group $\mathbf{R}$ *is defined to be* $qh(\mathbf{Z},\mathbf{Z})$ *modulo the bounded functions.*

Before proceeding to the multiplicative structure take $f \in qh(\mathbf{Z},\mathbf{Z})$ and suppose $|f(m + n) - f(m) - f(n)| \leqslant k$ for all $m, n \in \mathbf{Z}$. It is easy to deduce the inequality

$$|f(mn) - m\, f(n)| \leqslant (|m| + 1)k$$

and hence the inequality

$$(*) \qquad |n\, f(m) - m\, f(n)| \leqslant (|m| + |n| + 2)k.$$

[If we allow ourselves to know about rationals this implies $(f(n)/n)$ is a Cauchy sequence in $\mathbf{Q}$.]

It is easy to see that $qh(\mathbf{Z},\mathbf{Z})$ is closed under composition of functions $(f \circ g)(n) = f(g(n))$. This gives a multiplication on $qh(\mathbf{Z},\mathbf{Z})$ which almost makes it into a ring; all that fails is that $f \circ (g + h)$ and $f \circ g + f \circ h$ are not necessarily equal. However, by taking $m = g(n)$ in (*), we obtain an inequality

$$|n(f \circ g)(n) - g(n)\, f(n)| \leqslant (|n| + 1)k'$$

and hence an inequality

$$|(f \circ g)(n) - (g \circ f)(n)| \leqslant k''.$$

This shows that the multiplication of $qh(\mathbf{Z},\mathbf{Z})$ is almost commutative so that the failing distributive law is almost true as a consequence of the distributive law on the other side.

This proves that *composition in* $qh(\mathbf{Z},\mathbf{Z})$ *induces a multiplication on* $\mathbf{R}$ *which makes* $\mathbf{R}$ *a commutative ring.*

To see that $\mathbf{R}$ *is a field* take a quasi-homomorphism f which represents a non-zero element α of $\mathbf{R}$. Then f is not bounded above (or below). Let $\bar{f}(n)$ be the first integer m such that $f(m) \geqslant n$. Then $f(\bar{f}(n)) \geqslant n > f(\bar{f}(n) - 1) \geqslant f(\bar{f}(n) + f(1) - k$. So the difference between $f \circ \bar{f}$ and the identity function is bounded. It is easy to see that $\bar{f}$ is a quasi-homomorphism and so represents an inverse for α.

Call $\alpha \in \mathbf{R}$ *positive* when it is represented by a quasi-homomorphism f such that $f(n) \geqslant 0$ for all $n \geqslant 0$. Define $\alpha \leqslant \beta$ in $\mathbf{R}$ when $\beta - \alpha$ is positive. *This makes* $\mathbf{R}$ *an ordered field.*

To see that $\mathbf{R}$ *is order complete,* take a non-empty set S of positive elements of $\mathbf{R}$. For each $s \in S$, let $f_s \in qh(\mathbf{Z},\mathbf{Z})$ represent s and have $f_s(n) \geqslant 0$ for all $n \geqslant 0$. Define $g(n)$ to be the first element in the set $\{ f_s(n) | s \in S \}$ for $n \geqslant 0$. Put $g(n) = -g(-n)$ for $n < 0$. Then g represents a greatest lower bound for S.

Macquarie University

ANNOUNCEMENT: CAM NEWSLETTER

An international newsletter in computational and applied mathematics has been established by the editors of the *Journal of Computational and Applied Mathematics*. The first issue appeared (somewhat belatedly) in June 1984, and subsequent issues will appear three times a year.

The Newsletter will contain

- announcements of conferences, workshops and meetings;
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Information will be gathered from all over the world, except for the USA, for which this information is considered to be available from other sources.

I am the Australian coordinator for the Newsletter, and so would be happy to receive (at any time) any information for passing on to the editors.

A list of reports in an international newsletter provides valuable free advertising, so I urge departments to take advantage of this opportunity. For greatest efficiency it will be best to communicate directly with the editors:

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You should also write to the editors if you are interested in obtaining a copy of the Newsletter.

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