Abstract. If $A$ is an $n \times n$ complex matrix and $\lambda$ is an eigenvalue of $A$ with geometric multiplicity $k$, then $\lambda$ is in at least $k$ of the $n$ Geršgorin discs of $A$.

1. INTRODUCTION. One of the most attractive and useful results to locate the eigenvalues of a matrix is Geršgorin’s theorem, which goes back to 1931. The main part of the theorem is as follows.

Geršgorin Theorem. Let $A$ be an $n \times n$ real or complex matrix and let

$$R_i' = \sum_{j=1, j \neq i}^{n} |a_{ij}|, \quad \text{for } 1 \leq i \leq n$$

denote the deleted absolute row sums of $A$. Every eigenvalue of $A$ is located in the union of its $n$ Geršgorin discs

$$\bigcup_{i=1}^{n} D_i,$$

where

$$D_i = \{z \in \mathbb{C} : |z - a_{ii}| \leq R_i' \}.$$ 

The proof of the theorem involves a clever idea. Let $\lambda$ be an eigenvalue $A$, and suppose that

$$Ax = \lambda x, \ x = [x_i] \neq 0.$$ 

Some entry of $x$ has largest modulus, say $|x_p| \geq |x_i|$ for all $i = 1, 2, \ldots, n$, and $x_p \neq 0$. Then

$$x_p(\lambda - a_{pp}) = \sum_{j=1, j \neq p}^{n} a_{pj}x_j$$

and hence

$$|x_p| |\lambda - a_{pp}| \leq |x_p| \sum_{j=1, j \neq p}^{n} |a_{pj}| = |x_p| R_p'$$

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so that $|\lambda - a_{pp}| \leq R'_p$; that is, $\lambda$ lies in the $p$th Geršgorin disc. If $\lambda$ is a simple eigenvalue of $A$, that is the end of the story. However, if $\lambda$ is associated with several linearly independent eigenvectors, how could that fact be used to extract more information from Geršgorin’s theorem?

If the geometric multiplicity of $\lambda$, namely the dimension of the associated eigenspace (the null space of $\lambda I - A$), is 1, then we have no control over the position of a largest modulus entry of a corresponding eigenvector. Every eigenvector is a nonzero scalar multiple of some given eigenvector, so that every eigenvector has its largest modulus entry in the same position. However, if the geometric multiplicity of $\lambda$ is greater than 1, we have some flexibility in this regard. For example, let

$$A = \begin{bmatrix} 2 & 1 & 0 \\ -4 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$  \hspace{1cm} (1)

Then 0 is an eigenvalue of $A$ with (algebraic) multiplicity 3 and geometric multiplicity 2. The vectors $[1, -2, 3]^T$ and $[0, 0, 1]^T$ are linearly independent eigenvectors associated with $\lambda = 0$, with both largest modulus entries occurring in the third position. However, $[1, -2, 0]^T$ and $[0, 0, 1]^T$ are also eigenvectors, and their largest modulus entries occur in different positions. Our proof of Geršgorin’s theorem shows that the eigenvalue $\lambda = 0$ is in both the second and third Geršgorin discs; what we have observed here is no accident.

An $n \times n$ matrix $A$ has $n$ Geršgorin discs $D_i$, some of which may be duplicates, as in the trivial example of an identity matrix. We claim that if $\lambda$ is an eigenvalue of $A$ with geometric multiplicity $k$, then $\lambda$ is in at least $k$ of the Geršgorin discs of $A$.

2. RESULTS. To prove our claim, we need a key preliminary result: Each subspace $S$ of $C^n$ has a basis whose vectors have largest modulus entries in different positions. The argument uses a deflation process that has the same flavor as the proof in [2] of Schur’s triangularization theorem.

Lemma 2.1. Let $S$ be a $k$-dimensional subspace of $C^n$. There is a basis $\{v_1, v_2, \ldots, v_k\}$ of $S$ with the following property: For each $i = 1, 2, \ldots, k$, there are distinct integers $p_i$, with $1 \leq p_i \leq n$ and $p_i \neq p_j$ for $i \neq j$, such that a largest modulus entry of each $v_i$ is in position $p_i$.

Proof. We place the vectors of a basis $B = \{x_1, x_2, \ldots, x_k\}$ of $S$ as columns of an $n \times k$ full column rank matrix $X = [x_1 | \cdots | x_k]$. Let $P_1 \in M_n$ be a permutation matrix (not necessarily unique) such that a largest modulus entry of $x_1$ (there could be more than one) is the first entry of $P_1 x_1 = y_1$. Partition $P_1 X = [y_1 \, Y_2]$ and $y_1 = [y_{11} \, w^T]^T$. Let $R_1$ be an upper triangular matrix of the form

$$R_1 = \begin{bmatrix} 1 & \ast \\ 0 & I_{k-1} \end{bmatrix}$$

and choose the unique vector $z \in C^{n-1}$ such that

$$(P_1 X) R_1 = [y_1 \, Y_2] \begin{bmatrix} 1 & \ast \\ 0 & I_{k-1} \end{bmatrix} = [y_1 \, y_{11} z^* + Y_2] = \begin{bmatrix} y_{11} \, 0 \\ w \, X^{(2)} \end{bmatrix}.$$
has zero entries in the first row to the right of the (1, 1)-entry. Now, repeat this process on \(X^{(2)}, X^{(3)}, \ldots\) to obtain \((P_{k-1} \cdots P_1)X(R_1 \cdots R_{k-1}) = Z\), a lower-triangular matrix whose diagonal entries are largest modulus entries in their respective columns. Moreover, \(Z = PXR\), in which \(P\) is a product of \(k - 1\) permutation matrices and \(R\) is a product of \(k - 1\) upper-triangular matrices with 1s on the diagonal. Thus, \(P\) is a permutation matrix, \(R\) is upper-triangular and nonsingular, and \(Z\) has full column rank. Note that the column spaces of \(X\) and \(XR\) are the same. Thus, we see that the columns of \(P^T Z = XR\) have the desired property.

We can now use Lemma 2.1 to prove our claim. In the following discussion, \(A = [a_{ij}]\) is always an \(n \times n\) complex matrix.

**Theorem 2.2.** Let \(\lambda\) be an eigenvalue of \(A\) with geometric multiplicity \(k\). Then \(\lambda\) is in at least \(k\) of the Geršgorin discs \(D_i\) of \(A\).

**Proof.** Lemma 2.1 ensures that there is a basis \(\{x_1, x_2, \ldots, x_k\}\) of the eigenspace \(S\) of \(\lambda\) and distinct integers \(p_1, \ldots, p_k \in \{1, \ldots, n\}\) such that each vector \(x_i\) has a largest modulus entry in position \(p_i\). Our construction in the proof of the Geršgorin theorem shows that \(\lambda\) lies in Geršgorin discs \(D_{p_1}, \ldots, D_{p_k}\).

From Theorem 2.2, we see that an eigenvalue with geometric multiplicity at least \(k \geq 1\) is contained in any union of \(n - k + 1\) different Geršgorin discs of \(A\). Now that’s an improvement of Geršgorin’s general theorem, which is the case \(k = 1\) of our assertion!

**Corollary 2.3.** Let \(\lambda\) be an eigenvalue of \(A\) with geometric multiplicity at least \(k \geq 1\). Then

\[
\lambda \in \bigcup_{i=1}^{n-k+1} \{z \in C : |z - a_{ij}| \leq R'_{ij}\}
\]

for any choices of indices \(1 \leq i_1 < \cdots < i_{n-k+1} \leq n\). There are \(\binom{n}{k-1}\) possibilities for such a union, and \(\lambda\) is contained in their intersection.

The rank-nullity theorem says that for a matrix \(B\) with \(n\) columns, the rank of \(B\) plus the dimension of the null space of \(B\) is equal to \(n\). We apply this theorem and Theorem 2.2 to obtain the following result.

**Corollary 2.4.** Let \(\lambda\) be an eigenvalue of \(A\). If rank \((A - \lambda I) \leq t\), then \(\lambda\) is in at least \(n - t\) of the Geršgorin discs \(D_i\) of \(A\).

Using the same two theorems, we can prove another interesting and useful result.

**Corollary 2.5.** If \(|a_{ii}| > R'_i\) for \(q\) different values of \(i\), then the geometric multiplicity of \(\lambda = 0\) as an eigenvalue of \(A\) is at most \(n - q\) and rank \(A \geq q\).

The example (1) shows that the result in Theorem 2.2 is not valid for the algebraic multiplicity of an eigenvalue. Also, if \(\lambda\) is an eigenvalue of \(A\) with geometric multiplicity \(k\), then \(\lambda\) may be in more than \(k\) of the Geršgorin discs \(D_i\) of \(A\). For example, let

\[
A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}.
\]
Then 2 is an eigenvalue of \( A \) with geometric multiplicity 1 (and algebraic multiplicity 2), but the eigenvalue 2 is contained in two Geršgorin discs of \( A \). The same is true for the slightly more complicated example \( A_1 \), where

\[
A_1 = \begin{bmatrix}
3 & -\frac{1}{3} & 0 \\
2 & 1 & 0 \\
-1 & -1 & 3
\end{bmatrix},
\]

which has an additional eigenvalue of 3.

A final example shows that an eigenvalue \( \lambda \) can be in \( t \) Geršgorin discs, for some \( t \) between the geometric multiplicity of \( \lambda \) and the algebraic multiplicity of \( \lambda \). Let

\[
A_2 = \begin{bmatrix}
3 & 0 & -1 \\
-2 & 4 & 6 \\
1 & -1 & -1
\end{bmatrix}.
\]

Then 2 is an eigenvalue of \( A_2 \) with geometric multiplicity 1 and algebraic multiplicity 3; the eigenvalue 2 is contained in two Geršgorin discs of \( A_2 \). The direct sum matrix

\[
A_3 = \begin{bmatrix}
A_2 & 0 \\
0 & A_2
\end{bmatrix}
\]

has 2 as an eigenvalue with geometric multiplicity 2 and algebraic multiplicity 6; the eigenvalue 2 is contained in four Geršgorin discs of \( A_2 \).

Finally, we close with a question.

**Open Question 2.6.** Let \( k, r, t \) be positive integers with \( k \leq r \leq t \). Is there a square complex matrix \( A \) and an eigenvalue \( \lambda \) of \( A \) such that \( \lambda \) has geometric multiplicity \( k \) and algebraic multiplicity \( t \), and \( \lambda \) is in \( r \) Geršgorin discs of \( A \)?

After this paper was accepted for publication, the above open question was solved, see [1].

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