Chapter 1

Motivating properness

Perhaps the best known class of forcing notions is the class of ccc posets. A poset $\mathbb{P}$ is ccc iff there is no uncountable antichain of elements of $\mathbb{P}$. Among the $\omega_1$-preserving forcing notions, this is a nice class because it can be iterated while still preserving $\omega_1$, thus giving rise to the possibility of showing the consistency of a result for a class of uncountable objects by showing its consistency “one object at a time”.

This is how Suslin’s hypothesis was shown to be independent: $S$ is a Suslin tree iff it has height $\omega_1$, countable levels, and no chains or antichains of uncountable length. $S$, as a poset, is ccc, and in $V^S$, $S$ is no longer Suslin. By carefully killing these trees one at a time, we get a model of Suslin’s hypothesis, the assertion that there are no Suslin trees.

Forcing axioms are a natural development of this idea: We look at a class of $\omega_1$-preserving posets and prove a suitable iteration theorem. We then construct a model as a limit of an iteration where “every possible such poset” has been used, and therefore for any member of the class in the final model there is a partially generic filter (added along the iteration). In this model, we can obtain by purely combinatorial means results that traditionally would have been obtained using forcing.

MA comes from precisely this idea, by generalizing the iteration above so instead of just Suslin trees, all ccc posets are considered, see [26] and [38]. Strong forcing axioms are obtained by allowing not only ccc forcing notions but other posets as well. Nice classes for which this program can be implemented are:

- The class of proper posets.
- Semiproper posets.
- Stationary set preserving posets (although the reasons in this case are slightly different).
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1.1 Forcing notions that preserve $\omega_1$

It is easy to show, by iterating posets that destroy stationary sets (by shooting a club through their complement), that there are forcing notions that preserve $\omega_1$ but any iteration of them collapses it.

To present the example in more detail, we start by defining a poset $P_S$, for $S$ a stationary subset of $\omega_1$, such that $P_S$ preserves $\omega_1$ and shoots a club through $S$.

**Definition 1.1.1.** For $S$ a stationary subset of $\omega_1$ let

$$P_S := \{ p : \exists \alpha < \omega_1 (p : \alpha \rightarrow S), \ p \text{ is increasing, and } \ \text{ran}(p) \text{ is closed in } \omega_1 \}.$$  

The order is extension.

Notice that for any $p \in P_S$, dom $(p)$ is a successor ordinal.

**Lemma 1.1.2.** $P_S$ does not add any new countable subsets of $V$ (in particular, it preserves $\omega_1$), and adds a club subset of $S$.

**Proof.** We first prove that $P_S$ is $\omega_1$-distributive, i.e., $P_S$ does not add any new countable subsets of $V$. Since $P_S$ is separative, this is equivalent to asserting that the intersection of any $\omega$ many open dense sets is dense.

Let $(D_n : n \in \omega)$ be a sequence of dense open subsets of $P_S$, and let $p \in P_S$. Let $\lambda$ be a regular cardinal large enough so $P(P_S) \in \mathcal{H}_\lambda$. Let $X \prec \mathcal{H}_\lambda$ be countable and such that $P_S, p, (D_n)_{n \in \omega} \in X$ and $\delta \in S$, where $\delta = X \cap \omega_1$. $X$ exists, because $\mathcal{H}_\lambda$ is club in $[\mathcal{H}_\lambda]^\omega$, so $\{ N \cap \omega_1 : N \in \mathcal{H}_\lambda \}$ contains a club, and therefore meets $S$.

Let $(V_n)_{n \in \omega}$ enumerate all the open dense subsets of $P_S$ which belong to $X$, and define $(p_n)_{n \in \omega}$ so

- $p \geq p_0 \geq p_1 \geq \ldots$
- $p_n \in X \cap V_n$ for all $n$.

Let $q' = \bigcup_n p_n$ and set $q = q' \cup \{(\delta', \delta)\}$, where $\delta' = \text{dom}(q')$. Then $q \in P_S$, because

$$D^\alpha := \{ r \in P_S : \alpha < \sup \text{ran}(r) \}$$

is dense in $P_S$ for all $\alpha$, so $\sup \text{ran}(q') = \delta$.

By construction, since the $V_n$ are dense, $q \in \bigcap_n V_n \subseteq \bigcap_n D_n$ and $p \geq q$. Since $p$ is arbitrary, the result follows.

An immediate consequence of the $\omega_1$-distributivity of $P_S$ is that $\omega_1$ is preserved. Let $G$ be $P_S$-generic over $V$, and let $C = \bigcup \{ \text{ran}(p) : p \in G \} = \text{ran}(\bigcup G)$. Then $C$ is closed in its supremum, as each $p \in G$ is, and $C$ is unbounded in $\omega_1$ by considering the sets $D^\alpha$. This completes the proof.

As a final remark, notice that $\delta' = \delta$ in the notation used above. This is because $\{ p_n : n \in \omega \}$ generates a filter $g \overset{P}{\rightarrow}$-generic over $M$, where $M$ is the
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transitive collapse of $X$, and $\bar{P}$ is the image of $P_S$ under the collapse. Since $X \prec H_\lambda$

\[ M[g] \models \text{ran}(\bigcup g) \text{ is club in } \omega_1 = \omega_1^M, \]

so $\delta = \omega_1^M$ must equal $\text{dom}(\bigcup g) = \text{dom}(q') = \delta'$. □

**Corollary 1.1.3.** Any stationary subset of $\omega_1$ contains arbitrarily large countable closed subsets. □

The proof of Lemma 1.1.2 uses a device which reappears quite often in the theory of proper forcing.

**Definition 1.1.4.** Let $X$ be a countable model of enough set theory $T$ and let $P \in X$ be a forcing notion. A sequence $(p_n)_{n \in \omega}$ of conditions in $P$ is $X$-generic if and only if

- $p_0 \geq p_1 \geq \ldots$
- $p_n \in X$ for all $n$.
- For any open dense subset $D$ of $P$ which belongs to $X$ there is an $n$ such that $p_n \in D$.

Let $p \in \bar{P}$. An $X$-generic sequence $(p_n)_{n \in \omega}$ is below $p$ if and only if, in addition, $p \geq p_0$.

We need an additional property of $P_S$:

**Lemma 1.1.5.** Suppose $T \subset S$ is stationary. Then

\[ \Vdash_{P_S} T \text{ is stationary.} \]

**Proof.** Let $\dot{C}$ be any name for a club subset of $\omega_1$, let $q \in P_S$ and let $X \in H_\lambda$ be such that $\delta = X \cap \omega_1 \in T$, and $q, \dot{C}, P_S \in X$. For each $\alpha < \delta$,

\[ \V^\alpha = \{ p \in P_S : \exists \beta > \alpha (p \Vdash \beta \in \dot{C}) \} \]

is dense and belongs to $X$. If $r \in X \cap \V^\alpha$ then, by elementarity of $X$, some witness $\beta$ must exists inside $X$, i.e., $\exists \beta \in (\alpha, \delta)(r \Vdash \beta \in \dot{C})$.

Let $(p_n)_{n \in \omega}$ be an $X$-generic sequence below $q$, and let $p = \bigcup_n p_n \cup \{(\delta, \delta)\}$. As above, $p \in P_S$ and $q \geq p$. By the previous paragraph,

\[ p \Vdash \dot{C} \cap \delta \text{ is unbounded in } \delta. \]

Since $p \Vdash \dot{C}$ is club, this means that $p \Vdash \delta \in \dot{C}$.

Since $q$ is arbitrary, $\{ r \in P_S : r \Vdash \dot{C} \cap T \neq \emptyset \}$ is dense. Since $\dot{C}$ is arbitrary,

\[ \Vdash_{P_S} T \text{ is stationary,} \]

as we wanted to show. □
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Let \((S_n : n \in \omega)\) be a partition of \(\omega_1\) into \(\omega\) many stationary sets. Consider any iteration \((\mathbb{P}_n : i \leq \omega)\) where \(\mathbb{P}_0 = \mathbb{P}_{\omega_1 \setminus S_0}\) and \(\mathbb{P}_{n+1} \equiv \mathbb{P}_n \ast \mathbb{P}_{\omega_1 \setminus S_{n+1}}\) for each \(n \in \omega\). Then each \(\mathbb{P}_n\) preserves \(\omega_1\), because in fact each \(S_k, k > n\), remains stationary after forcing with \(\mathbb{P}_n\). On the other hand, \(\mathbb{P}_\omega\) collapses \(\omega_1\), no matter what kind of limit is considered, because if \(C_n\) is the \(n\)th club set added by the forcing, then \(\bigcap_n C_n = \emptyset\).

This also shows that an obstacle to overcome if we want an iteration theorem for a class of \(\omega_1\) preserving forcing notions is that stationary subsets of \(\omega_1\) may be destroyed by forcing notions in the class.

1.2 Preserving stationary subsets of \(\omega_1\)

So we may want to look at posets that preserve stationary sets. This is not enough, however, as examples can be given of posets that preserve stationary subsets of \(\omega_1\), whose iteration collapses \(\omega_1\).

Remark 1.2.1. Much can be said about this class of posets. For example, the corresponding forcing axiom (MM) is consistent and as strong as possible (see Chapter 2). However, this is due to the fact that SPFA, the forcing axiom for a more restrictive class is consistent, namely the class of semiproper forcing notions (see Section 1.4), and this forcing axiom implies that every stationary set preserving poset is semiproper. This is not always the case; for example, the statement implies \(0^\#\), and more importantly, semiproper forcing notions can be iterated but, as we are about to show, stationary set preserving posets cannot. See Sections 1.4 and 2.2 for additional details.

To define the example, let us first introduce two orderings on \(^{\omega_1}\omega_1\).

Definition 1.2.2. Let \(f, g : \omega_1 \rightarrow \omega_1\).

- \(f \prec_{NS_{\omega_1}} g\) if and only if \(\{ \alpha : f(\alpha) < g(\alpha) \}\) contains a club.
- \(f \prec^* g\) if and only if \(\exists \alpha \forall \beta > \alpha (f(\beta) < g(\beta))\).

Clearly, \(f \prec^* g\) implies \(f \prec_{NS_{\omega_1}} g\), and \(<_{NS_{\omega_1}}\) is well-founded. The example, due to Shelah, consists on generically adding an infinite \(\prec_{NS_{\omega_1}}\)-descending sequence.

The following is immediate:

Claim 1.2.3. If \(A \subseteq ^{\omega_1} \omega_1\) and \(|A| \leq \aleph_1\), then \(\exists g \forall f \in A (f \prec^* g)\). \(\square\)

For technical reasons, we need CH to hold in the ground model. By considering Col\((\omega, \aleph_1)\), this requirement is no loss of generality:

Lemma 1.2.4. Let \(\mathbb{P}\) be a \(\sigma\)-closed forcing. Then \(\mathbb{P}\) preserves stationary subsets of \(\omega_1\).
Recall that \( P \) is \( \kappa \)-closed iff for every \( \beta < \kappa \) and every decreasing sequence \( (p_\alpha : \alpha < \beta) \) of elements of \( P \) there is in \( P \) a lower bound for the sequence. \( P \) is \( \sigma \)-closed iff it is \( \omega_1 \)-closed.

**Proof.** Let \( S \) be stationary in \( \omega_1 \), \( \dot{C} \) a name for a club subset of \( \omega_1 \), \( p \in P \) and let \( \lambda \) be a sufficiently large regular cardinal. Pick \( X \in \dot{H}_\lambda \) with \( p, P, C \subseteq X \) and \( \delta = X \cap \omega_1 \in S \). Let \( (p_\alpha)_{\alpha \in \omega} \) be \( X \)-generic below \( p \), and let \( q \) be a lower bound for it. \( q \) exists by the \( \sigma \)-closure of \( P \). Then (exactly as in the proof of Lemma 1.1.5) \( q \Vdash \delta \in \dot{C} \).

By density, \( \Vdash_P S \) is stationary. \( \square \)

So we may assume \( CH \) holds.

By Claim 1.2.3, there is a \( <^* \)-increasing sequence \( \vec{f} = (f_\alpha : \alpha < \omega_2) \). First we define a forcing \( P \) that adds an upper bound to \( \vec{f} \).

**Remark 1.2.5.** It is consistent that any \( <^* \)-increasing sequence of length \( \omega_2 \) is actually cofinal in \( (\omega_1, \leq^*) \), so this first step is necessary in general.

To see this, assume for example that the nonstationary ideal \( NS_{\omega_1} \) on \( \omega_1 \) is saturated, so if \( G \) is \( P(\omega_1)/NS_{\omega_1} \)-generic over \( V \), then in \( V[G] \) there is an elementary embedding

\[
j : V \rightarrow V^{\omega_1}/G
\]

with critical point \( \omega_1 \) and such that \( j(\omega_1) = \omega_2 \). Suppose that \( f <^* g \). Then obviously \( [f] < [g] < \omega_2 \) in \( V^{\omega_1}/G \), where \( [f] := j(f)(\omega_1) \), and any \( <^* \)-increasing sequence of length \( \omega_2 \) is necessarily cofinal in \( (\omega_1, \leq^*) \). Larson and Shelah \[25\] have produced an example where any \( <^* \)-increasing sequence of length \( \omega_2 \) is cofinal, and \( CH \) holds.

**Definition 1.2.6.** \( P = \omega_1 \times ^{\omega_1} \omega_1 \). Order \( P \) by \( (\alpha, f) < (\beta, g) \) if and only if \( \alpha \geq \beta, f|\beta = g|\beta, \) and \( g <^* f \).

\( P \) is \( \sigma \)-closed, so it preserves stationary subsets of \( \omega_1 \). Let \( G \) be \( P \)-generic over \( V \), and set \( f_G = \bigcup \{ g|\alpha : (\alpha, g) \in G \} \). Then \( f_G : \omega_1 \rightarrow \omega_1 \) and \( h <^* f_G \) for any \( h \in V \).

**Claim 1.2.7 (CH).** \( P \) preserves \( \omega_2 \).

**Proof.** We actually show that \( P \) is \( \omega_2 \)-cc and, in fact, that it satisfies a natural version of the Knaster property at \( \omega_2 \): Any family of \( \omega_2 \)-many conditions contains a family of size \( \omega_2 \) of pairwise compatible conditions. The argument is similar to that for Solovay’s almost-disjoint forcing, but we require \( CH \): Let \( \{ p_\gamma : \gamma < \omega_2 \} \) be a family of \( \omega_2 \) many conditions in \( P \). We show that they do not form an antichain.

By the pigeonhole principle, we may assume that there is a fixed \( \alpha \in \omega_1 \) such that \( (p_\gamma)_0 = \alpha \) for all \( \gamma \). Since \( \text{Card}^{\omega_1} \leq \aleph_1 = \aleph_1 \), we may as well assume that all the conditions share the same stem, i.e., there is a fixed \( f : \alpha \rightarrow \omega_1 \) such that \( (p_\gamma)_1 | \alpha = f \) for all \( \gamma \). But then all the \( p_\gamma \) are compatible. \( \square \)

Let \( f = f_G \). We are in the following situation: With the obvious meaning of the notation, \( \vec{f} <_{NS_{\omega_1}} f \) (in fact, \( \vec{f} <^* f \)). Given any \( g \) such that \( \vec{f} <_{NS_{\omega_1}} g \) we define a forcing \( P_g \) such that
\[ \mathbb{P}_g \text{ preserves stationary subsets of } \omega_1. \]

\[ \mathbb{P}_g \text{ preserves } \omega_2 \text{ (at least, under CH).} \]

\[ \mathbb{P}_g \text{ is } \omega_1\text{-distributive (so, if CH holds, it is preserved).} \]

\[ \mathbb{P}_g \text{ adds a generic function } h : \omega_1 \to \omega_1 \text{ such that } f^\ast <_{\text{NS}_{\omega_1}} h <_{\text{NS}_{\omega_1}} g. \]

But then, since it introduces an infinite \(<_{\text{NS}_{\omega_1}}\)-descending sequence, any iteration \((\mathbb{P}_i : i \leq \omega)\) collapses \(\omega_1\), where \(\mathbb{P}_0 = \mathbb{P}_f\) and \(\mathbb{P}_{n+1} \cong \mathbb{P}_n \ast \mathbb{P}_{\dot{g}_n}\), for \(\dot{g}_n\) a \(\mathbb{P}_n\)-name for its generic function.

**Definition 1.2.8.** Let \(f^\ast\) be as above and let \(g : \omega_1 \to \omega_1\) be such that
\[
\forall \alpha < \omega_2 \, (f^\ast <_{\text{NS}_{\omega_1}} g)
\]
(in short, \(f^\ast <_{\text{NS}_{\omega_1}} g\)). Let \(\mathbb{P}_g\) be the set of all \(p = (h^p, s^p, F^p, T^p)\) such that

- For some \(\alpha < \omega_1\), \(h^p : \alpha + 1 \to \omega_1\). We write \(\alpha + 1 = \text{dom}(p)\).
- \(s^p\) is the characteristic function of a closed subset of \(\alpha + 1\); for all \(\beta < \alpha + 1\), \(s^p(\beta) = 1\) implies \(h^p(\beta) < g(\beta)\).
- \(F^p \in [\omega_2]^{\leq \omega}\).
- \(T^p : F^p \to \{\text{closed subsets of } \alpha + 1\}\); for all \(\gamma \in F^p\) and \(\beta \in T^p(\gamma)\), \(f^\ast(\beta) < h^p(\beta)\).

The function that \(\mathbb{P}_g\) is intended to add is \(h = \bigcup_{p \in G} h^p\), where \(G\) is generic. \(\mathbb{P}_g\) also adds club sets intended to witness \(h\) has the desired properties: \(s = \bigcup_{p \in G} s^p\) is the characteristic function of a club witnessing \(h <_{\text{NS}_{\omega_1}} g\). For \(\gamma < \omega_2\), \(\bigcup\{T^p(\gamma) : p \in G, \gamma \in F^p\}\) is a club witnessing \(f^\ast <_{\text{NS}_{\omega_1}} h\).

We define the ordering of \(\mathbb{P}_g\) so this is actually the case:

**Definition 1.2.9.** Order \(\mathbb{P}_g\) by: \(p < q\) iff

- \(h^p \supseteq h^q\),
- \(s^p \supseteq s^q\),
- \(F^p \supseteq F^q\), and
- \(T^p(\gamma) \cap \text{dom}(q) = T^q(\gamma)\) for all \(\gamma \in F^q\).

Note that for any \(\alpha < \omega_1\) and any \(p \in \mathbb{P}_g\), \(p\) can be extended to a condition \(q\) such that \(\beta + 1 := \text{dom}(q) > \alpha\), \(s^q(\beta) = 1\) and \(\beta \in \bigcap T^p(F^p)\). This ensures (once we check that \(\mathbb{P}_g\) preserves \(\omega_1\)) that if \(G\) is \(\mathbb{P}_g\)-generic over \(V\), then \(s = \bigcup_{p \in G} s^p\) has indeed domain \(\omega_1\) and for all \(\gamma < \omega_2\), \(C_\gamma = \bigcup\{T^p(\gamma) : p \in G, \gamma \in F^p\}\) is indeed unbounded.

We now verify that \(\mathbb{P}_g\) satisfies the requirements stated above.

**Claim 1.2.10 (CH).** \(\mathbb{P}_g\) is \(\omega_2\)-cc and therefore preserves \(\omega_2\).
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**Proof.** We show that $\mathbb{P}_g$ is $\omega_2$-Knaster. Start as in Claim 1.2.7: Let $\{p_\gamma : \gamma < \omega_2\}$ be a collection of $\omega_2$-many conditions in $\mathbb{P}_g$. We show they do not form an antichain. First, by CH, we can assume for some fixed $\alpha < \omega_1, h: \alpha + 1 \to \omega_1$ and $s: \alpha + 1 \to 2$, that for all $\gamma < \omega_2$, dom $(p_\gamma) = \alpha + 1$, $h^{p_\gamma} = h$ and $s^{p_\gamma} = s$.

Second, by the $\Delta$-system lemma (see [21, Theorem II.1.6]), we can as well assume that the $F^{p_\gamma}$ form a $\Delta$-system (applying CH once again)

1. Let $\Delta$ be its root. For each $\beta \in \Delta$ there are only $R_1$ possible values for $T^{p_\gamma}(\beta)$. Since $\Delta$ is countable, we can assume the value of $T^{p_\gamma}|\Delta$ is fixed. But then all the $p_\gamma$ are compatible. $\square$

**Theorem 1.2.11.** $\mathbb{P}_g$ preserves stationary subsets of $\omega_1$ and is $\omega_1$-distributive.

**Proof.** Let $S \subset \omega_1$ be stationary, $p \in \mathbb{P}_g, \dot{C}$ a name for a club, and $\dot{r}$ a name for a function from $\omega$ into $V$. It is enough to find some $q \leq p$, $\delta \in S$ and $r: \omega \to V$ such that $q \Vdash (\delta \in \dot{C} \text{ and } \dot{r} = r)$.

To do this, let $\lambda$ be large (so, in particular, $\Vdash_{\mathbb{P}_g} \dot{r}: \omega \to H_\lambda$) and regular, and let $N \prec H_\lambda$ be such that $\mathbb{P}, p, S, \dot{C}, \dot{r} \in N$, $\omega_1 + 1 \subset N$, and $|N| = \omega_1$. Let $\varepsilon = N \cap \omega_2$, so $\varepsilon$ is an ordinal. Let $\zeta: \omega_1 \forces \varepsilon$.

Let $N = ((N_\alpha, \zeta(\omega_1 \cap N_\alpha) : \alpha < \omega_1)$ be a continuous increasing sequence of countable elementary substructures of $(N, \zeta)$ such that $\mathbb{P}_g$-preserves stationary subsets of $\omega_1$:

$N_\alpha = \bigcup_{\beta < \alpha} N_\beta$ whenever $\alpha$ is limit. Notice that $\zeta \notin N$.

For $\alpha < \omega_1$, let $\pi_\alpha: N_\alpha \to (M_\alpha, \check{\zeta})$ be the transitive collapse of $N_\alpha$. Then $\zeta: \omega_1 \forces \varepsilon$.

We define some club subsets of $\omega_1$:

- Let $C_0$ be a club such that $\alpha \in C_0$ implies $\alpha = N_\alpha \cap \omega_1$. Then, for all $\alpha \in C_0$, $N_\alpha \cap \varepsilon = \zeta[\alpha]$.
  (It is in order to define $C_0$ that we needed to preserve $\omega_2$.)

- Let $C_1$ be a club such that $\alpha \in C_1$ implies $f_\varepsilon(\alpha) < _{NS_{\omega_1}} g$.

- For $\alpha < \omega_1$, let $C^\alpha = [\gamma_\alpha, \omega_1)$, where $\gamma_\alpha$ is such that $\beta \geq \gamma_\alpha$ implies $f_{\zeta(\alpha)}(\beta) < f_\varepsilon(\beta)$.

- Finally, let $C = C_0 \cap C_1 \cap \Delta_\alpha C^\alpha$.

Let $\delta \in C \cap S$. Let $(p_n)_{n \in \omega}$ be $N_\delta$-generic below $p$. Define $q \in \mathbb{P}_g$ by:

1. $\text{dom}(h^q) = \delta + 1, h^q|\delta = \bigcup_n h^{p_n}$ and $h^q(\delta) = f_\varepsilon(\delta)$.

Notice that $h^q$ is well-defined: Since $(p_n)_{n \in \omega}$ is $N_\delta$-generic,

$$\sup_{n} \text{dom}(p_n) = \delta.$$
O is well-defined because $\delta \in C_1$, so $h^q(\delta) = f_\xi(\delta) < g(\delta)$.

(3) $F^q = \bigcup_n F^p_n$.

(4) For $\gamma \in F^q$, let $n_\gamma$ be least such that $\gamma \in F^p_{n_\gamma}$, and set

$$T^q(\gamma) = \bigcup_{n \geq n_\gamma} T^{p_n}(\gamma) \cup \{\delta\}.$$ We need to show that $\gamma \in F^q$ implies $f_\gamma(\delta) < h^q(\delta)$.

To see this, notice $F^q \subseteq N_\delta$, so $\gamma = \zeta(\mu)$ for some $\mu < \delta$, since $\delta \in C_0$. Then $f_\gamma(\delta) = f_\xi(\mu)(\delta)$. But since $\delta \in \bigtriangleup_n C^\alpha$, $\delta \in C^\mu$, so $f_\xi(\mu)(\delta) < f_\xi(\delta) = h^q(\delta)$.

Hence, $q \in P_g$. Since $(p_n)_{n \in \omega}$ is $N_\delta$-generic, $q \Vdash \check{C} \cap \delta$ is unbounded in $\delta$, so $q \Vdash \delta \in \check{C}$. Similarly, for all $n$ there is an $m_n$ and an $r_n \in N_\delta$ such that $p_{m_n} \Vdash r(n) = r_n$. Let $r : \omega \to V$ be the function $r(n) = r_n$. Then $q \Vdash r = r$. This completes the proof. \qedsymbol

### 1.3 Properness

So we need a strengthening of “preserving stationary sets”, and here we present Shelah’s suggestion, which is responsible for most of what follows.

**Definition 1.3.1** (Shelah). $\mathbb{P}$ is proper iff for all uncountable sets $X$ and all stationary $S \subseteq [X]^\omega$, $V^\mathbb{P} \models S$ is stationary.

Properness can be characterized in terms of a genericity condition involving the structures $\mathcal{H}_\theta$ and their substructures, and it is this characterization that we will use in what follows.

**Definition 1.3.2.** Let $\theta$ be regular and let $\mathbb{P} \in \mathcal{H}_\theta$. Let $N \in \check{H}_\theta$ and suppose that $\mathbb{P} \in N$. A condition $q \in \mathbb{P}$ is $(N, \mathbb{P})$-generic iff for every maximal antichain $D \subseteq \mathbb{P}$ such that $D \in N$, $D \cap N$ is predense below $q$, i.e., for any $r \leq q$ there is an $s \in D \cap N$ compatible with $r$.

**Theorem 1.3.3** (Shelah). $\mathbb{P}$ is proper iff for every regular

$$\theta > |\{D \subseteq \mathbb{P} : D \text{ is a maximal antichain}\}|,$$

the set

$$\{N \in \check{H}_\theta : \mathbb{P} \in N \text{ and } \forall p \in \mathbb{P} \cap N \exists q \leq p (q \text{ is } (N, \mathbb{P})\text{-generic})\}$$

contains a club in $[\mathcal{H}_\theta]^\omega$.

**Proof.** ($\Rightarrow$) Clearly $C = \{N \in \check{H}_\theta : \mathbb{P} \in N\}$ is club. For $p \in \mathbb{P}$, let $S_p = \{N \in C : p \in N \text{ and no } q \leq p \text{ is } (N, \mathbb{P})\text{-generic}\}$.
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**Claim 1.3.4.** $S_p$ is nonstationary.

Granting the claim, we are done: Let

$$S = \{ N \in C : \exists p \in N (N \in S_p) \}.$$ 

Then $S$ is nonstationary, by Fodor’s lemma, and $C \setminus S$ is precisely the set we need to show contains a club.

**Proof of Claim 1.3.4.** Let $G_p$ be $\mathbb{P}$-generic over $V$ with $p \in \mathbb{P}$. Work in $V[G_p]$ and define $f$ from the set of maximal antichains of $\mathbb{P}$ in $V$ to $G_p$ by setting 

\[ \{ f(A) \} = G_p \cap A \text{ for any such antichain } A. \]

Let $C_p = \{ N \in [\mathcal{K}_p^0]^{\omega} : \mathbb{P} \in N \prec \mathcal{K}_p^0 \text{ and } \forall D \in N \cap V \text{ maximal antichain of } \mathbb{P} (f(D) \in N) \}.$

Then $C_p$ is club but $C_p \cap S_p = \emptyset$. For if $N \in S_p$ and $q \leq p$, there is a maximal antichain $D \in N$ that is not predense below $q$, so there is $q' \leq q$ such that $q'$ is incompatible with every element of $D \cap N$. It follows that the set of such $q'$ (which clearly belongs to $V$) is dense below $p$, so there is such a $q' \in G_p$.

Suppose $N \in C_p$. Let $D$ be a maximal antichain of $\mathbb{P}$ in $N$ such that all the elements of $D \cap N$ are incompatible with $q'$. Then $f(D) \in G_p \cap N \cap D$, and therefore $q'$ and $f(D)$ are compatible (since they are in $G_p$) and incompatible (by choice of $D$), a contradiction.

It follows that $S_p$ is not stationary in $V[G_p]$. But $\mathbb{P}$ is proper, so $S_p$ cannot be stationary in $V$. \[ \square \]

$(\Leftarrow)$ Let $X$ be uncountable and let $S \subseteq [X]^{\omega}$ be stationary. We show that if $p_0 \Vdash \dot{f} : [X]^{<\omega} \rightarrow X$ then there is $q \leq p_0$ and $z \in S$ such that $q \Vdash z$ is closed under $\dot{f}$. By a density argument, it follows that $S$ is stationary in $V^{\mathbb{P}}$, so $\mathbb{P}$ is proper.

Let $\theta$ be regular and sufficiently large, so

\[ \{ N \in H_\theta : \mathbb{P}, \dot{f}, X, p_0 \in N \text{ and } \forall p \in \mathbb{P} \cap N \exists q \leq p \text{ $(N, \mathbb{P})$-generic} \} \]

contains a club $C$. Then $\{ N \cap X : N \in C \}$ contains a club in $[X]^{\omega}$. Since $S$ is stationary, there is $N \in C$ such that $N \cap X \in S$.

Let $q_0 \leq p_0$ be $(N, \mathbb{P})$-generic.

**Claim 1.3.5.** $q_0 \Vdash N \cap X$ is closed under $\dot{f}$.

**Proof.** Let $x \in [N \cap X]^{<\omega}$. Then $x \in N$, so

$$\mathcal{D} = \{ q \in \mathbb{P} : q \leq p_0 \text{ and } \exists a \in X (q \Vdash \dot{f}(x) = a) \} \in N.$$ 

Notice that $\mathcal{D}$ is open dense below $p_0$. Let $E \in N$ be a maximal antichain below $p_0$ contained in $\mathcal{D}$.

Let $q' \leq q_0$ be arbitrary such that for some $a \in X$ it holds that $q' \Vdash \dot{f}(x) = a$. Since $q_0$ is $(N, \mathbb{P})$-generic, $E \cap N$ is predense below $q_0$, so there is an $r \in E \cap N$ compatible with $q'$. It follows that $r \Vdash \dot{f}(x) = a$, and since $r, \dot{f} \in N$, then
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$a \in N$. Hence, $q' \Vdash \check{f}(x) \in N \cap X$. It follows (since $q'$ was arbitrary deciding $\check{f}(x)$) that the same is forced by $q_0$. □

This completes the proof. □

To illustrate the way this characterization is used, let us show the following easy result.

Fact 1.3.6. If $\mathbb{P}$ is ccc or $\omega_1$-closed, then $\mathbb{P}$ is proper.

Proof. Let $\theta$ be regular and sufficiently large, let $p \in \mathbb{P}$ and let $p, \mathbb{P} \in N \in \mathcal{H}_\theta$.

Suppose first that $\mathbb{P}$ is ccc. We claim that $p$ itself is $(N, \mathbb{P})$-generic. In effect, let $D \in N$ be a maximal antichain of $\mathbb{P}$. Since $\mathbb{P}$ is ccc, $D$ is countable, so $D \subset N$ and clearly $D \cap N = D$ is predense below $p$.

Suppose now that $\mathbb{P}$ is $\omega_1$-closed. Let $(p_n)_{n \in \omega}$ be $N$-generic below $p$, and let $q \leq p_n$ for each $n$. Then $q$ is clearly $(N, \mathbb{P})$-generic. □

A similar argument shows that any Axiom A forcing notion is proper.

Definition 1.3.7. $\mathbb{P}$ is an Axiom A forcing poset as witnessed by $\langle \leq_n : n \in \omega \rangle$ iff

1. $\leq_0 \subset \leq_1$,
2. $\leq_{n+1} \subset \leq_n$ for each $n \in \omega$,
3. whenever $(p_n)_{n \in \omega}$ is a fusion sequence:

$$p_0 \geq_0 p_1 \geq_1 \cdots \geq_n p_{n+1} \geq_{n+1} \cdots,$$

there is $q$ such that for all $n$, $q \leq_n p_n$, and
4. for all $p \in \mathbb{P}$, $n \in \omega$, and $D \subset \mathbb{P}$ dense, there are a countable set $D' \subset D$ and a condition $q \leq_n p$ such that $q \Vdash D' \cap G \neq \emptyset$.

Lemma 1.3.9 below generalizes Fact 1.3.6 in view of the following.

Fact 1.3.8. ccc and $\sigma$-closed forcings are Axiom A forcings.

Proof. For $\mathbb{P}$ ccc, for all $n$ let $\leq_n$ be the identity. Given $p \in \mathbb{P}$ and $D \subset \mathbb{P}$ dense, any maximal antichain below $p$ can be taken as $D'$ in clause 4 of the definition of Axiom A.

For $\mathbb{P}$ $\sigma$-closed, let $\leq_n = \leq$ for all $n$. □

Lemma 1.3.9. Axiom A forcing posets are proper.

Proof. Let $\mathbb{P}$ be an Axiom A forcing, and let $S \subset [A]^\omega$ be stationary. For $p \in \mathbb{P}$ and $\check{f}$ such that $p \Vdash \check{f} : [A]^\omega \to A$, we find $q \leq p$ and $a \in S$ such that $q \Vdash a \in \text{cl} \check{f}$.

The proof is routine: Let $\lambda$ be regular and sufficiently large, and pick $N \in \mathcal{H}_\lambda$ such that $\mathbb{P}, p, A, S, \check{f} \in N$ and $N \cap A \in S$. Let $(D_n)_{n \in \omega}$ enumerate all dense subsets of $\mathbb{P}$ belonging to $N$, and inductively define a fusion sequence $(p_n)_{n \in \omega}$ and a sequence of countable sets $(D'_n : n \in \omega)$ such that
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1. \( D'_n \subseteq D_n \).
2. \( p \geq p_0 \).
3. \( p_n \models D'_n \cap G \neq \emptyset \) for each \( n \).
4. \( p_n, D'_n \in N \) for each \( n \).

Let \( q \leq p_n \) for each \( n \). Since the \( D'_n \) are countable, \( \bigcup_n D'_n \subseteq N \), so \( q \models N \cap A \in \text{cl}_f \). \( \square \)

Most forcing notions for adding reals (e.g., Sacks forcing, Mathias forcing, Laver forcing) are Axiom A forcing notions. Notice also the following.

**Fact 1.3.10.** If \( P \) is an Axiom A poset, and \( \models_P Q \) is an Axiom A poset, then \( P \ast Q \) is an Axiom A poset. \( \square \)