Determinacy And Jonsson Cardinals

**Definition.** $\kappa$ is a *Jonsson cardinal* if for every countable algebra $\{f_i\}_{i \in \omega}$ (each $f_i : \kappa^n \to \kappa$ for some $n$) there is a proper subalgebra of size $\kappa$.

That is, an $A \subseteq \kappa$ with $|A| = \kappa$, $A \neq \kappa$, and $A$ closed under the $f_i$.

note: can use a single $f : \kappa^{<\omega} \to \kappa$.

**Fact (ZF).** Measurable $\Rightarrow$ Ramsey $\Rightarrow$ Rowbottom $\Rightarrow$ Jonsson

**Corollary (AD + $V = L(\mathbb{R})$).** Every regular $\kappa < \Theta$ is Jonsson. (by Steel)

**Fact (ZF).** If there exists a Jonsson cardinal, then $0\#$ exists. So, in $L$ there are no Jonsson cardinals.

It is consistent with ZFC that every Jonsson cardinal is Ramsey, and it is also consistent that there is a Jonsson cardinal of cofinality $\omega$. 
Fact (Tryba, Woodin). Assume ZFC. If \( \kappa \)

is a regular Jonsson cardinal then every sta-

tionary \( S \subseteq \kappa \) reflects.

In particular, successors of regular card-

inals are not Jonsson.

Fact (ZFC). The least Jonsson cardinal is
either weakly inaccessible or of cofinality \( \omega \).

We work in ZF + DC + AD.

Theorem 1. Assume AD. Every uncount-
able cardinal \( \kappa < \aleph_{\epsilon_0} \) (or even \( \kappa < \aleph_{\omega_1} \))
is Jonsson.

In fact, the proof also shows:

Theorem 2. Assume AD. Every uncount-
able cardinal \( \kappa < \aleph_{\epsilon_0} \) (or even \( \kappa < \aleph_{\omega_1} \))
is Rowbottom.

Conjecture. Assume AD. Then every \( \kappa < \Theta \) is Jonsson.

[note added after talk: Woodin showed this conjecture follows from the directed sys-
tem analysis of HOD. It is not clear if one can get Rowbotton.)
Recall $\aleph_\epsilon_0$ is the supremum of the projective ordinals.

Also, $\aleph_{\omega_1}$ is the supremum of the $\delta^1_\alpha$ for $\alpha < \omega_1$.

The proof uses a “linear” analysis of the cardinal structure below the projective ordinals, not just the “non-linear” description analysis. “Linear” refers to representing the cardinals as $j_\mu(\delta^1_{2n+1})$, where $\mu$ is a measure on $\delta^1_{2n+1}$.

Brief History:

• (mid 80’s) J develops description analysis of cardinal structure below projective ordinals (and a ways beyond). This computes the $\delta^1_{2n+1}$ and shows they have the strong partition property. Lower bound follows from following theorem of Martin:

Theorem 3 (Martin). Assume $\kappa \rightarrow (\kappa)^\kappa$. Let $\mu$ be a measure on $\kappa$. Then $j_\mu(\kappa)$ is a cardinal.
• (mid 90’s) J and Khafizov connect the linear and description analyses. In particular, every description gives a cardinal. Analysis gives algorithm for converting between two representations of the cardinals.

• J and Löwe give general framework for the linear analysis; ordinal algebras.

We recall some facts about the projective ordinals and the description analysis.

• \( \delta_1^1 = \aleph_1, \delta_3^1 = \aleph_{\omega+1}, \delta_5^1 = \aleph_{\omega \cdot \omega+1}, \)
• \( \delta_{2n+1}^1 = \aleph_{w(2n-1)+1}, \) where \( w(1) = \omega \) and \( w(n+1) = \omega^{w(n)}. \)
• \( \delta_{2n+2}^1 = (\delta_{2n+1}^1)^+. \)
• There are \( 2^{n+1} - 1 \) regular cardinals strictly between \( \delta_{2n+1}^1 \) and \( \delta_{2n+3}^1. \)

The cardinals between \( \delta_{2n+1}^1 \) and \( \delta_{2n+3}^1 \) are of the form

\[
\kappa = (\text{id}; W_{2n+1}^m; d; K_1, \ldots, K_t),
\]

where \( d \) is a description, \( W_{2n+1}^m \) is a canonical measure on \( \delta_{2n+1}^1 \), and \( K_1, \ldots, K_t \) are
canonical measures on \( \lambda_{2n+1} = (\delta^1_{2n+1})^{-} \). Here id is the identity function from \( \delta^1_{2n+1} \) to \( \delta^1_{2n+1} \).
Canonical Measures

$W^m_1$ is the $m$-fold product of the normal measure on $\omega_1$.

$S^m_1$ is the measure on $\omega_{m+1}$ induced by the strong partition relation on $\omega_1$, functions $f : \text{dom}(<_m) \to \omega_1$ of the correct type, and the measure $W^m_1$. Here,

$$(\alpha_1, \ldots, \alpha_m) <_m (\beta_1, \ldots, \beta_m) \leftrightarrow$$

$$(\alpha_m, \alpha_1, \ldots, \alpha_{m-1}) <_{\text{lex}} (\beta_m, \beta_1, \ldots, \beta_{m-1})$$

$W^m_3$ is the measure on $\delta^1_3$ induced by the weak partition relation on $\delta^1_3$, functions $f : \omega_{m+1} \to \delta^1_3$ of the correct type and the measure $S^m_1$.

In general, there is a family corresponding to each regular cardinal.

$$S^{1,m}_{2n+1}, \ldots, S^{2n+1,m}_{2n+1}, W^m_{2n+3}$$

$S^{i,j}_{2n+1}$ a measure on $\lambda_{2n+3}$;

$W_{2n+3}$ a measure on $\delta^1_{2n+3}$.

$S^{1,m}_{2n+1}$ is measure defined using strong partition relation on $\delta^1_{2n+1}$, ordering $<_m$ on $(\delta^1_{2n+1})^m$. 
and the $m$-fold product of the $\omega$-cofinal normal measure on $\delta_{2n+1}^1$.

$S_{2n+1}^{i,m}, i \geq 2,$ is measure defined using strong partition relation on $\delta_{2n+1}^1$, and measure $\mu^{i,m}$ on $\delta_{2n+1}^1$. $\mu^{i,m}$ is measure defined from the weak partition relation on $\delta_{2n+1}^1$, function $f: \text{dom}(\nu^{i,m}) \rightarrow \delta_{2n+1}^1$, and $\nu^{i,m} = \text{the } m\text{th measure in the } i-1\text{st family in list of canonical families: }$

$W_1, S_1, W_3, S_3^1, S_3^2, S_3^3, W_5, \ldots$

[For $i = 2$ we identify $W_1^m$ with a measure on $\omega_1$ by reverse lexicographic ordering on the $m$-tuples $(\alpha_1, \ldots, \alpha_m).$]
**Ordinal Algebras**

**Definition.** An *ordinal algebra* is a free algebra on a set of generators

\[ \mathcal{A}_\alpha = \{ v_\beta : \beta < \alpha \} \]

under the operations \( \oplus, \otimes \).

We define the *height* of a term \( t \in \mathcal{A} \) as follows:

\[
\begin{align*}
o(v_0) &= 0 \\
o(s \oplus t) &= o(s) + o(t) \\
o(s \otimes t) &= o(s) \cdot o(t) \\
o(v_\alpha) &= \sup\{o(t) + 1 : t \in \mathcal{A}_\alpha \}
\end{align*}
\]

**Examples**

\((\alpha = 2)\) \( \mathcal{A}_2 = \langle v_0, v_1 \rangle. \)

\(o(v_0) = 0, o(v_1) = 1, o(v_1 \oplus \cdots \oplus v_1) = n.\)

\(\text{ht}(\mathcal{A}_2) = \omega.\)

\((\alpha = 3)\) \( \mathcal{A}_3 = \langle v_0, v_1, v_2 \rangle. \)

\(o(v_0) = 0, o(v_1) = 1, o(v_2) = \omega.\)
\[ o(v_2 \otimes \cdots \otimes v_2) = \omega^n \]

\[ \text{ht}(A_3) = \omega^\omega \]

\[ (\alpha = \omega) \ A = \langle v_n : n \in \omega \rangle. \]

\[ o(v_n) = \omega^{\omega n-2} \ (n \geq 3). \]

\[ \text{ht}(A_\omega) = \omega^{\omega^\omega}. \]

In general: \[ \text{ht}(A_\alpha) = \begin{cases} 
1 & \alpha = 1 \\
\omega^{\omega n-2} & \text{if } \alpha < \omega \\
\omega^{\omega \alpha} & \text{if } \alpha \geq \omega 
\end{cases} \]

We identify terms \( t \in A \) with labelled trees (labelled by the basic variables).

\[ v_1 \oplus ((v_1 \oplus v_2) \otimes v_2) \oplus (v_4 \otimes v_3) \]

corresponds to

![Diagram](image-url)
We assign measures to the terms $t \in \mathcal{A}$ (canonical measure assignment).

$\nu_0 \leftrightarrow \text{“empty measure”}$.
$\nu_1 \leftrightarrow \text{principal measure}.$
$\nu_2 \leftrightarrow W_1^1 = \text{normal measure on } \omega_1.$
$\nu_3 \leftrightarrow S_1^1 = \text{the } \omega\text{-cofinal normal measure on } \omega_2.$
$\nu_n \leftrightarrow S_1^{n-2}.$

(that’s all we need below $\delta_5^1$)

$\nu_\omega \leftrightarrow \text{the } \omega\text{-cofinal normal measure on } \delta_3^1.$
$\nu_\omega \cdot 2 \leftrightarrow \text{the } \omega\text{-cofinal normal measure on } \mathfrak{N}_{\omega \cdot 2 + 1}.$
$\nu_\omega \omega \leftrightarrow \text{the } \omega\text{-cofinal normal measure on } \mathfrak{N}_{\omega \omega}.$

We will have that:

$\nu_\omega \leftrightarrow \text{the } \omega\text{-cofinal normal measure on } \delta_3^1$

$\nu_{\omega \omega} \leftrightarrow \text{the } \omega\text{-cofinal normal measure on } \delta_5^1$
\( \nu_w(5) \leftrightarrow \) the \( \omega \)-cofinal normal measure on \( \delta_1^{-1} \), etc.

In general, suppose we have assigned the measure \( \nu(t) \) to every basic variable in \( \mathcal{A}_{w(2n-1)} \), and these are all measure on \( \lambda_{2n+1} \).

Consider a general term \( t \in \mathcal{A}_{w(2n-1)} \). To \( t \) is associated an order type, namely lexicographic ordering on tuples \( \langle i_0, \alpha_0, \ldots, i_k, \alpha_k \rangle \) where \( (i_0, \ldots, i_k) \) is in the tree associated to \( t \) and \( \alpha_j \in \text{dom}(\nu(v(i_0, \ldots, i_j))) \).

For example, for the term

\[
t = v_1 \oplus ((v_1 \oplus v_2) \otimes v_2) \oplus (v_4 \otimes v_3)
\]

considered before \( (t \in \mathcal{A}_\omega) \), with corresponding tree
the order-type is $1 + (1 + \omega) \times \omega + \omega^{\omega^2} \cdot \omega^\omega = \omega^{\omega^2} \cdot \omega^\omega$.

We usually view $\text{ot}(t)$ as the set of tuples $\langle i_0, \alpha_0, \ldots, i_k, \alpha_k \rangle$ In this case, if $i_0 = 0$, then $\alpha_0 = 0$, if $i_0 = 1$, then $\alpha_0 < \omega_1$, and if $i_0 = 2$, then $\alpha_0 < \omega_2$, etc.

We refer to this indexed collection of measures as the germ associated to $t$, $g = g(t)$.

Given $t$ and associated germ $g$, suppose $f : \text{ot}(t) \to \delta_{2n+1}^1$ is order-preserving. For each $\vec{i} = (i_0, \ldots, i_k) \in \text{dom}(g)$, $f$ induces a subfunction

$$f^\vec{i} : \nu(i_0) \times \nu(i_0, i_1) \times \cdots \times \nu(i_0, \ldots, i_k) \to \delta_{2n+1}^1$$

by

$$f^\vec{i}(\alpha_0, \ldots, \alpha_k) = f(\langle i_0, \alpha_0, \ldots, i_k, \alpha_k \rangle).$$
By $[f^i]$ we mean the equivalence class with respect to this product measure.

**Definition.** \(\text{wlift}(t)\) is the measure on \((\delta_{2n+1})^{<\omega}\) generated by the weak partition relation on \(\delta_{2n+1}\), functions \(f: ot(t) \to \delta_{2n+1}\) of the correct type, and the product measures \(\nu(i_0) \times \cdots \times \nu(i_0, \cdots i_k)\).

We identify \(\text{wlift}(t)\) with a measure on \(\delta_{2n+1}\) by a fixed ordering on the tuples, say reverse KB ordering.

**Definition.** \(\text{slift}(t)\) is the measure on \(\lambda_{2n+3}\) generated by the strong partition relation on \(\delta_{2n+1}\), functions \(F: \delta_{2n+1} \to \delta_{2n+1}\), and the measure \(\text{wlift}(t)\) on \(\delta_{2n+1}\).

To \(v_{2+\omega(2n-1)+o(t)}\) we associate the measure \(\text{slift}(t)\).

**Canonicity Assumption:** Let \(\mu = \text{wlift}(t)\). Then \(j_\mu(\delta_{2n+1}) = \aleph_{\omega(2n-1)+o(t)+1}\).
This embodies the connection between the cardinal structure given by the canonical measure assignment and the structure given by the descriptions.

We make this assumption from now on.

We define a projection operator $\pi$ which we apply to the measures $\nu(t)$ associated to $t \in A$.

We define $\pi$ on the first few levels directly for simplicity.

$$\pi(v_0) = v_0, \pi(v_1) = v_1, \pi(v_2) = v_2, \pi(v_3) = v_3$$

(i.e., $\pi(W_1^1) = W_1^1$, $\pi(S_1^1) = S_1^1$).

$$\pi(v_{n+2}) = \pi(S_{1}^{n+1}) = \tilde{S}_1^n = \text{the measure induced by the strong partition relation on } \omega_1, \text{functions } f: <_n \to \omega_1 \text{ which are order-preserving and } f(\alpha_1, \ldots, \alpha_n) \text{ has uniform cofinality } \alpha_n \text{ (and the measure } W_1^n).$$
For $t \in \mathcal{A}$ with corresponding germ $g \approx \nu^{\langle i_0, \ldots, i_k \rangle}$, let $\vec{j} = \langle j_0, \ldots, j_l \rangle$ be the least significant index (so this is of maximal length in the tree).

If $\pi(\nu^{\vec{j}}) = \nu^{\vec{j}}$, let $g'$ be $g$ restricted to the domain $\text{dom}(g) - \{(j_0, \ldots, j_l)\}$. Otherwise, let $\text{dom}(g') = \text{dom}(g)$, and let $g' = g$ except $g'(\vec{j}) = \pi(\nu^{\vec{j}})$. Then $\pi(g)$ is the measure on $\delta_{2n+1}^1$ corresponding to the germ $g'$ and functions $f' : \text{ot}(g') \rightarrow \delta_{2n+1}^1$ of type induced by $f : \text{ot}(g) \rightarrow \delta_{2n+1}^1$.

Define then $\pi(\nu_{2+w(2n-1)+o(t)})$ to be the measure induced by the strong partition relation on $\delta_{2n+1}^1$, functions $F : \delta_{2n+1}^1 \rightarrow \delta_{2n+1}^1$, and the measure $\pi(g)$ on $\delta_{2n+1}^1$.

Let $\mathcal{M} = \{\nu(t) : t \in \mathcal{A}\}$, and $\tilde{\mathcal{M}} = \{\pi^k(\mu(t)) : t \in \mathcal{A}\}$.

So, $\mathcal{M} \subseteq \tilde{\mathcal{M}}$. 
We consider the case of successor cardinals $\kappa$ between $\delta_3^1$ and $\delta_5^1$.

Fix $t \in A_\omega$ and corresponding measure $\nu = \nu(t) = \nu^j$ such that $\kappa = j_\nu(\delta_3^1)$.

Example: If $\kappa = \aleph_\omega^2 \cdot \aleph_\omega + 1$, let $t = \nu_4 \otimes \nu_3$. 
\[ \text{ot}(g) = \aleph_3 \cdot \aleph_2. \]
Lemma 1. Suppose $f, g : \omega_1 \to \delta^1_3$ are increasing. Then either:

(1) $\sup(f) < \sup(g)$ (or vice-versa). In this case there are $f', g' : \omega_1 \to \delta^1_3$ with $[f']_{W_1^1} = [f]_{W_1^1}$, $[g']_{W_1^1} = [g]_{W_1^1}$, $\text{ran}(f') \subseteq \text{ran}(f)$, $\text{ran}(g') \subseteq \text{ran}(g)$, and $f(\alpha) < g(\beta)$ for all $\alpha, \beta < \omega_1$.

or

(2) $\sup(f) = \sup(g)$. Then there are $f', g'$ with $[f']_{W_1^1} = [f]_{W_1^1}$, $[g']_{W_1^1} = [g]_{W_1^1}$, $\text{ran}(f') \subseteq \text{ran}(f)$, $\text{ran}(g') \subseteq \text{ran}(g)$, and $f(\alpha) < g(\alpha) < f(\alpha + 1)$ for all $\alpha < \omega_1$ (or vice-versa).

Proof. In case (1), define $f' = f$ and $g'(\alpha) = g(\alpha_0 + \alpha)$ where $\alpha_0$ is least such that $g(\alpha_0) > \sup(f)$.

In case (2), let $C \subseteq \omega_1$ be c.u.b. such that for $\alpha, \beta$ in $C$, $f(\alpha) < g(\alpha) < f(\beta)$. Define

\[ f'(\alpha) = f(\alpha^{th} \text{element of } C') \]
\[ g'(\alpha) = g(\alpha^{th} \text{element of } C') \]
This lemma easily generalizes to the following.

For $f: <_{n} \rightarrow \omega_{1}$ order-preserving and $i \leq n$, define $f^{i}: (\omega_{1})^{i} \rightarrow \omega_{1}$ by

$$f^{i}(\alpha_{1}, \ldots, \alpha_{i}) = \text{sup}_{\vec{\beta}} f(\alpha_{1}, \ldots, \alpha_{i-1}, \vec{\beta}, \alpha_{i})$$

For $\alpha < \omega)_{n+1}$ represented by $f: <_{n} \rightarrow \omega_{1}$ order-preserving, let also $\alpha^{i} = [f^{i}]_{W_{1}}$. This is easily well-defined.

**Lemma 2.** Let $f, g: <_{n} \rightarrow \omega_{1}$ be order-preserving, $[f^{i-1}] = [g^{i-1}]$, and $[f^{i}] = [g^{i}]$. Then there are $f'$, $g'$ with $[f'] = [f]$, $[g'] = [g]$, $\text{ran}(f') \subseteq \text{ran}(f)$, $\text{ran}(g') \subseteq \text{ran}(g)$, and for all $\vec{\alpha}, \vec{\beta}$, $f(\vec{\alpha}) < g(\vec{\beta})$ iff

$$(\alpha_{n}, \alpha_{1}, \ldots, \alpha_{i}) \leq_{n} (\beta_{n}, \beta_{1}, \ldots, \beta_{i}).$$

Next we consider functions $f: \text{dom}(S_{1}^{n}) \rightarrow \delta_{1}^{1}$. 
If \( f : \text{dom}(S_1^n) \to \delta_3^1 \), let \( f^0 = \sup(f) \), and for \( i \leq n \) let \( f^i : \text{dom}(S_1^i) \to \delta_3^1 \) be defined by:

\[
f^i(\alpha) = \sup\{f(\beta) : \beta^i = \alpha\}.
\]

**Lemma 3.** Let \( f, g : \text{dom}(S_1^n) \to \delta_3^1 \) be increasing. Let \( i \leq n \) be least such that \([f^i] \neq [g^i]\). Then there are \( f' \), \( g' \) with \([f'] = [f]\), \([g'] = [g]\), \( \text{ran}(f') \subseteq \text{ran}(f)\), \( \text{ran}(g') \subseteq \text{ran}(g)\), and for all \( \alpha, \beta \in \text{dom}(S_1^n) \) we have: \( f'(\alpha) < g'(\beta) \) iff \( (\alpha^i \leq \beta^i) \).

**Proof.** Fix \( f, g : \text{dom}(S_1^n) \to \delta_3^1 \) increasing and \( i \leq n \) with \([f^i] < [g^i]\) and \([f^{i-1}] = [g^{i-1}]\). Consider the partition \( \mathcal{P} \):

We partition \( h_1, h_2 : <n \to \omega_1 \), of the correct type and ordered by \( h_1(\vec{\alpha}) < h_2(\vec{\beta}) \) iff \( (\alpha_n, \alpha_1, \ldots, \alpha_i) < (\beta_n, \beta_1, \ldots, \beta_i) \). (i.e., \( h_1^i = h_2^i \) and \( h_2^{i+1} < h_1^{i+1} \)), according to whether

\[
f([h_1]) < g([h_2]).
\]
Easily, on the homogeneous side the stated property holds. Let \( C \subseteq \omega_1 \) be homogeneous for \( \mathcal{P} \).

For \( \beta = [h] \in \text{dom}(S^n_1) \), let \( \beta' \) be represented by

\[
(\alpha_1, \ldots, \alpha_n) \mapsto h(l_C(\alpha_1), \ldots, l_C(\alpha_n)),
\]
where \( l_C(\alpha) = \alpha^{th} \) element of \( C \).

Define

\[
f'(\beta) = f(\beta') \quad \quad g'(\beta) = g(\beta')
\]

This easily works, using the previous lemma.

This generalizes to finitely many functions as well. We also generalize to functions from \( \text{ot}(g) \to \delta^1_3 \).

**Lemma 4.** Let \( t \in A_\omega \) with corresponding germ \( g = \nu^{(i_0, \ldots, i_k)} \). Let

\[
f_1, \ldots, f_n : \text{ot}(g) \to \delta^1_3
\]
be increasing. Then there are \( f'_1, \ldots, f'_n \) with \([f'_k] = [f_k] \), \( \text{ran}(f'_k) \subseteq f_k \), and for
all $1 \leq a, b \leq n$, all $\vec{i} = \langle i_0, \ldots, i_k \rangle$, $\vec{j} = \langle j_0, \ldots, j_l \rangle \in \text{dom}(g)$, and all $(\alpha_0, \ldots, \alpha_k)$, $(\beta_0, \ldots, \beta_l)$ we have for some $m, p$ that

$$f_a^{\vec{i}}(\alpha_0, \ldots, \alpha_k) < f_b^{\vec{j}}(\beta_0, \ldots, \beta_l)$$

iff

$$(i_0, \alpha_0, \ldots, i_m, \alpha_m^p) \leq (j_0, \beta_0, \ldots, j_m, \beta_m^p).$$

(or likewise ending with an integer).

If $\bar{\alpha} < \delta_3^1 = [f]$ for $f : \text{ot}(g) \to \delta_3^1$, let $l$ be least such that $\pi^l(\alpha) = \pi^\infty(\alpha)$. Let $\bar{\alpha}^i = \pi^{l-i}(\bar{\alpha})$.

Consider now a function $F : \text{dom}(\nu(g)) \to \delta_3^1$ (recall $\text{dom}(\nu(g)) = \delta_3^1$).

Let $F^i(\bar{\alpha}) = \sup\{F(\bar{\beta}) : \bar{\beta}^{i-1} = \bar{\alpha}\}$. 
Lemma 5. Let $F_1, \ldots, F_n: \delta_3^1 \to \delta_3^1$ be increasing. Then there are $F'_1, \ldots, F'_n$ with $[F_i]_{\mu} = [F_i]_{\mu}$, $\text{ran}(F'_i) \subseteq \text{ran}(F_i)$, and such that for all $1 \leq a, b \leq n$, there is a $p$ such that for all $\vec{\alpha}, \vec{\beta} \in \text{dom}(\mu(g))$ we have

$$F'_a(\vec{\alpha}) < F'_b(\vec{\beta})$$

iff

$$\vec{\alpha}^p \leq \vec{\beta}^p.$$  

Proof. Consider the case of two functions $F, G$. Let $i$ be least such that $[F^i]_{\pi^{l+1-i}(\mu)} < [G^i]_{\pi^{l+1-i}(\mu)}$.

Consider the partition

$\mathcal{P}$: partition pairs $f, g: \text{dom}(\text{ot}(g)) \to \delta_3^1$ each of the correct type with $f^i = g^i$ and with $g$ smaller otherwise, according to whether

$$F([f]_{\nu}) < G([g]_{\nu}).$$

Easily, on the homogeneous side the stated property holds. Fix a c.u.b. $C \subseteq \delta_3^1$ homogeneous for $\mathcal{P}$. 
Define

\[ F'(\langle f \rangle) = F(\langle f' \rangle) \]

where \( f'(\vec{\alpha}) = f(\vec{\alpha})^{th} \) element of \( C \). Likewise define \( G' \). This works using the previous lemma.

\[ \square \]

proof that \( \kappa \) is Jonsson

Fix the algebra \( \ell: \kappa^{<\omega} \to \kappa \).

By previous lemma, for every \( n \) and every \( F_1, \ldots, F_n: \text{dom}(\mu) \to \delta^1_3 \), there is a finitary \( \mathcal{M} \) which describes the arrangement of the \( F_i \).

For each \( n + 1 \) and \( \mathcal{M} \), consider the partition \( \mathcal{P}(n, \mathcal{M}) \) where we partition \( F_1, \ldots, F_n, G \) of type \( \mathcal{M} \) (all distinct) according to whether \( \ell([F_1], \ldots, [F_n]) \neq [G] \).

On the homogeneous side the stated property holds (using \( [G] \neq [F_i] \)).
Let $C \subseteq \delta_3^1$ be homogeneous for all these partitions. Let $D = C'$.

Clearly $A = \{ [F]_{\mu} : F : \delta_3^1 \to D \text{ of correct type } \}$ has size $\kappa$.

But, $\ell''(A^{<\kappa}) \neq \kappa$ as it contains no $[G]_{\mu}$ where $G : \delta_3^1 \to (C - D)$. (for such a $G$, $[G]_{\mu} \neq [F_i]_{\mu}$). This follows by the previous lemma and the homogeneity of $C$. 