THE FOURTEEN VICTORIA DELFINO PROBLEMS AND THEIR STATUS IN THE YEAR 2015

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§1. Introduction. The Victoria Delfino Problems played an important role in the development of descriptive set theory in the context of the Cabal. The first set of problems (# 1 to # 5) were announced during one of the *Very Informal Gatherings of Logicians* (VIG) at UCLA in 1978. They were subsequently published as an Appendix [KM78A] in [Cabal i] with the following explanations and rules:

> The following list of problems was distributed during a very informal gathering of logicians at UCLA in January 1978. We are reproducing it here because of its obvious relevance to the contents of this volume.
>
> A cash prize of $100 is offered by the logicians in the Los Angeles area for the solution of each of the following five problems. This competition is financed by the Victoria Delfino Fund for the Advancement of Logic which was established by a generous contribution from Miss Victoria Delfino.
>
> Employees of UCLA and Caltech and their immediate families (other than students) are ineligible for these prizes; competition is open to everyone else. All decisions by the judges are final. Multiple entries are allowed.

1.1. Victoria Delfino. Most of this section is based on recollections shared by Yiannis Moschovakis with the first author during a telephone conversation.

Victoria Delfino was the realtor in the Los Angeles area who helped Moschovakis buy his house. When Tony Martin moved to UCLA, Moschovakis referred him to Ms. Delfino, who also became his realtor and found the house where Martin still lives. Two weeks after the sale was finalized, Victoria Delfino gave Moschovakis a relatively large amount of money as commission for the referral, and did not accept his attempts to reject it.

As a result, Moschovakis decided instead to use the money to help fund the series of VIGs, the first of which had taken place in the Fall of 1974. The second VIG, in 1978, started a new tradition as well, and with a single exception, they have all taken place on Super Bowl weekend, in late January or early February. As Moschovakis told the first author,

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> The time of the only exception, there was an earthquake! A clear sign that moving the date was a mistake.

Together with the funding of the VIGs, the money was also set to cover the prizes for the solutions of the five original Delfino Problems. (Contrary to popular belief, no monetary prize was attached to further problems.)

When Moschovakis introduced these five problems (in what he described as one of the most significant VIGs to date), and mentioned the *Victoria Delfino Fund*, Martin, taken by surprised, exclaimed "That's my broker!" Not all in attendance heard this, and Moschovakis offered no further explanation on who Ms. Delfino was our why the Fund was named after her. This led for a short while to a variety of conjectures trying to find appropriate interpretations to explain the name.

Originally, the Fund was kept in a joint account by Alexander Kechris, Martin, and Moschovakis. It was supplemented by occasional donations from other logicians in the area. Eventually, it became so low that it made sense to use it all on the current VIG. After John Steel solved problem # 12, it was briefly discussed at the 1998 VIG whether the Fund still existed. The answer was inconclusive. As Martin explained to the first author:

> The reason it was "inconclusive" whether the Victoria Delfino Fund existed in 1998 was that by that time all the money in the Fund had been used and we had stopped asking people to contribute to it.

The account was finally closed. Nowadays, the VIG meetings are typically funded through support of the NSF.

As for Ms. Delfino, she eventually retired, moved out of state to take care of an ill relative, and her trail disappears there. It is not clear whether she ever found out that her name was associated with the problems or with the Cabal.

1.2. The problems. After the first announcement of the Victoria Delfino Problems, progress reports were published in [Cabal ii] and [Cabal iii]. In 1985, three of the original problems had been solved, and seven new problems (# 6 to # 12) were added and published as an Appendix [KMS88A] in [Cabal iv], preceded by the following:

> At the "Very Informal Gathering" of January 1984, the Cabal announced the addition of seven problems to the Victoria Delfino list. We are happy (and not at all embarrassed) to report that since then four of these problems have been solved. Below we list the new problems, beginning with # 6 since there were five problems on the original list. For each we describe briefly what was known when it was added to the list, and what has been its fate since.

In the years following the publication of the final original Cabal volume, there were two more problems announced at one of the VIGs in the late 1980s or early 1990s (the precise date could not be identified), but they were never published as Victoria Delfino Problems. Our statement of these two questions is not a direct quotation from that announcement.

Today, two of the problems remain open. The first one is better known under the name of *Martin's Conjecture* (# 5), the other one has now been embedded into Woodin's theory AD^+ (# 14). In Table 1, the reader can find a synoptic list of the problems with their current status.

	Published in	Status
# 1	[KM78A]	Solved by Steve Jackson (1983)
# 2	[KM78A]	Solved by Yiannis Moschovakis (1981)
# 3	[KM78A]	Solved by Howard Becker & Alexander Kechris (1983)
# 4	[KM78A]	Solved by John Steel (1993)
# 5	[KM78A]	Open
# 6	[KMS88A]	Solved by John Steel (1984)
# 7	[KMS88A]	Solved by Steve Jackson (1985)
# 8	[KMS88A]	Solved by John Steel (1994)
# 9	[KMS88A]	Solved by Tony Martin & John Steel (1985)
# 10	[KMS88A]	Solved by W. Hugh Woodin & Saharon Shelah (1985)
# 11	[KMS88A]	Solved by John Steel (1994)
# 12	[KMS88A]	Solved by John Steel (1997)
# 13	unpublished	Solved by W. Hugh Woodin (1999)
# 14	unpublished	Open

TABLE 1. List of the Victoria Delfino Problems and their current status

This paper is organized as follows: Each problem is presented in its own section that, except for the last two problems, starts with a quote from the appendix to the original Cabal volume where the problem was originally stated (in a subsection that cites the relevant appendix and is titled "Original problem"). The quotation is essentially literal, although we have updated notation, fixed obvious typographical errors, and replaced symbols with prose when the change improves readability. For the first five problems, this is followed by one or several subsections titled "Progress report" where we reproduce the text from subsequent appendices providing updates on the problem. We then proceed with a brief discussion of the current state of knowledge in a subsection titled "2015 comments".

1.3. Acknowledgements. We want to thank Daisuke Ikegami, John Steel, Simon Thomas, Hugh Woodin, and Yizheng Zhu, for detailed comments and remarks that we have incorporated in what follows whenever possible. Special thanks are due to Antonio Montalbán, Jan Reimann, and Ted Slaman, for sharing their results on Martin's conjecture (some of their remarks are not included since they are discussed in [MSS14]), to Ralf Schindler, for his comments and corrections on problem # 12, and for informing us of [SUW14], and to Yiannis Moschovakis and Tony Martin for their recollections on Victoria Delfino, the Delfino Fund, and the early history of the VIGs.

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1. Projective Ordinals.

Original problem [KM78A].

For each positive integer n, let $\utilde{\boldsymbol{\delta}}^1_n$ be the least nonzero ordinal not the length of a $\utilde{\boldsymbol{\Delta}}^1_n$ prewellordering of the reals.

Assume $\mathsf{AD}+\mathsf{DC}$. It is known that $\utilde{\boldsymbol{\delta}}^1_1 = \omega_1$, $\utilde{\boldsymbol{\delta}}^1_2 = \omega_2$, $\utilde{\boldsymbol{\delta}}^1_3 = \omega_{\omega+1}$, $\utilde{\boldsymbol{\delta}}^1_4 = \omega_{\omega+2}$, $\utilde{\boldsymbol{\delta}}^1_{2n+2} = (\utilde{\boldsymbol{\delta}}^1_{2n+1})^+$, and $\utilde{\boldsymbol{\delta}}^1_{2n+1}$ is always the successor (cardinal) of a cardinal of cofinality ω.

Problem # 1. *Compute* $\utilde{\boldsymbol{\delta}}^1_5$.

Kunen has some partial results on this problem, results which suggest the answer ω_{ω^3+1}.

The problem is related to that of whether $\utilde{\boldsymbol{\delta}}^1_3 \to (\utilde{\boldsymbol{\delta}}^1_3)^{\utilde{\boldsymbol{\delta}}^1_3}$. Kunen has shown that $\utilde{\boldsymbol{\delta}}^1_3 \to (\utilde{\boldsymbol{\delta}}^1_3)^\alpha$ for each $\alpha < \utilde{\boldsymbol{\delta}}^1_3$. Results of Kleinberg imply that $\utilde{\boldsymbol{\delta}}^1_3$ has exactly three normal measures. It is likely that the regular cardinals between $\utilde{\boldsymbol{\delta}}^1_3$ and $\utilde{\boldsymbol{\delta}}^1_5$ are exactly the ultrapowers of $\utilde{\boldsymbol{\delta}}^1_3$ with respect to these normal measures. This would be important in getting an upper bound on $\utilde{\boldsymbol{\delta}}^1_5$ from Choice plus $\mathsf{AD}^{\mathbf{L}(\mathbb{R})}$, the hypothesis that every set of reals in $\mathbf{L}(\mathbb{R})$ is determined.

(Needless to say, the decision of the judges as to what constitutes a "computation" of $\utilde{\boldsymbol{\delta}}^1_5$ will be final.)

Progress report [KMM81A].

Martin has established the conjectured lower bound for $\utilde{\boldsymbol{\delta}}^1_5$ by proving (from $\mathsf{AD}+\mathsf{DC}$) that

$$\utilde{\boldsymbol{\delta}}^1_5 \geq \aleph_{\omega^3+1};$$

moreover Martin showed (from AD) that the ultrapowers of $\utilde{\boldsymbol{\delta}}^1_3 = \aleph_{\omega+1}$ under the three normal measures on $\utilde{\boldsymbol{\delta}}^1_3$ are exactly $\utilde{\boldsymbol{\delta}}^1_4 = \aleph_{\omega+2}$ (this was known to Kunen), $\aleph_{\omega\cdot 2+1}$ and $\aleph_{\omega^2+1}$ and that these three cardinals are measurable (and hence regular), so that (in particular), $\utilde{\boldsymbol{\delta}}^1_5$ is not the first regular cardinal after $\utilde{\boldsymbol{\delta}}^1_4$. We still have no upper bounds for $\utilde{\boldsymbol{\delta}}^1_5$ from AD.

Progress report [KMM83A].

It was announced in [KMM81A] that Martin had shown $\utilde{\boldsymbol{\delta}}^1_5 \geq \aleph_{\omega^3+1}$ and that the ultrapowers of $\utilde{\boldsymbol{\delta}}^1_3$ with respect to the three normal measures on $\utilde{\boldsymbol{\delta}}^1_3$ are $\aleph_{\omega+2}$, $\aleph_{\omega\cdot 2+1}$ and $\aleph_{\omega^2+1}$. The proof of part of the last assertion, that the ultrapower by the ω_2-cofinal measure is $\leq \aleph_{\omega^2+1}$, was incorrect. Actually this ultrapower is larger ($\aleph_{\omega^\omega+1}$).

Steve Jackson has completely solved the first problem. He first proved that $\underset{\sim}{\boldsymbol{\delta}}^1_5 \leq \aleph_{\omega^{(\omega^\omega)}+1}$. This result will appear in his UCLA Ph.D. Thesis. He next used the machinery for getting this upper bound to analyze all measures on $\underset{\sim}{\boldsymbol{\delta}}^1_3$ and to get a good representation of functions with respect to these measures. Martin observed that this representation and ideas of Kunen allow one to show $\underset{\sim}{\boldsymbol{\delta}}^1_3 \to (\underset{\sim}{\boldsymbol{\delta}}^1_3)^{\underset{\sim}{\boldsymbol{\delta}}^1_3}$. From this it follows by a result of Martin that the ultrapower of $\underset{\sim}{\boldsymbol{\delta}}^1_3$ with respect to any of its measures is a cardinal. Jackson's analysis then gives $\underset{\sim}{\boldsymbol{\delta}}^1_5 \geq \aleph_{\omega^{(\omega^\omega)}+1}$ so $\underset{\sim}{\boldsymbol{\delta}}^1_5 = \aleph_{\omega^{(\omega^\omega)}+1}$.

2015 comments.

Steve Jackson not only solved problem # 1, but also solved the problem in general for all projective ordinals. He computed $\underset{\sim}{\boldsymbol{\delta}}^1_{2n+1}$ to be $\aleph_{\mathbf{e}_n+1}$ where $\mathbf{e}_0 := 0$ and $\mathbf{e}_{i+1} := \omega^{\omega^{\mathbf{e}_i}}$ (*i.e.*, $\mathbf{e}_n$ is an exponential ω-tower of height $2n-1$).

However, Jackson's paper [Jac88] (reprinted in [Cabal II]), where the inequality $\underset{\sim}{\boldsymbol{\delta}}^1_{2n+1} \leq \aleph_{\mathbf{e}_n+1}$ is established, is notoriously hard to read, and so in the decades following his solution of the problem, Jackson produced various expositions of the results. As the title suggests, his book [Jac99] focuses on the (complete) computation of $\underset{\sim}{\boldsymbol{\delta}}^1_5$ as asked in the original problem, and explains how to proceed to compute all projective ordinals via an inductive analysis. His survey paper [Jac10] in the Handbook [KF10] also discusses some of the extensions of the results for the projective ordinals. The following quote is from page 1755 of [Jac10]:

> In the early 1980s, Martin [MarA] obtained a result on the ultrapowers of $\underset{\sim}{\boldsymbol{\delta}}^1_3$ by the normal measures on $\underset{\sim}{\boldsymbol{\delta}}^1_3$. Building on this and some joint work with Martin, the author computed $\underset{\sim}{\boldsymbol{\delta}}^1_5$. In the mid-1980s, this was extended to compute all the $\underset{\sim}{\boldsymbol{\delta}}^1_n$, and to develop the combinatorics of the cardinal structure of the cardinals up to that point. The analysis, naturally, proceeded by induction. The complete "first-step" of the induction appears in [Jac99]. The analysis revealed a rich combinatorial structure to these cardinals. [. . .] A goal, then, is to extend some version of this "very-fine" structure theory to the entire model $\mathbf{L}(\mathbb{R})$. In the late 1980s the author extended the analysis further, up to the least inaccessible cardinal in $\mathbf{L}(\mathbb{R})$, although this lengthy analysis has never been written up. It was clear, however, that new, serious problems were being encountered shortly past the least inaccessible. In [Jac91], for example, results were given that show that the theory fell far short of $\kappa^{\mathbb{R}}$, the ordinal of the inductive sets (the Wadge ordinal of the least non-selfdual pointclass closed under real quantification).

The survey [Jac10] closes with a discussion of the "weak square property" $\boxminus_{\kappa,\lambda}$ (κ a Suslin cardinal, λ an ordinal less than Θ), a combinatorial principle part of a program by Jackson to extend the theory in a global way that does not depend on inductive arguments.

#2. The extent of definable scales.

Original problem [KM78A].

A **semiscale** on a set $P \subseteq \mathbb{R}^k$ ($\mathbb{R} = {}^\omega\omega$) is a sequence $\vec{\phi} = \{\phi_n\}_{n\in\omega}$ of **norms** on P, where each $\phi_n : P \to \lambda$ maps P into some ordinal λ and the following convergence condition holds: If $x_0, x_1, \ldots \in P$ and for each n the sequence $\phi_n(x_0), \phi_n(x_1), \phi_n(x_2), \ldots$ is ultimately constant, then $x \in P$. We call $\vec{\phi}$ a **scale** if, under the same hypotheses, we can infer that

$$\phi_n(x) \le \phi_n(x_i) \text{ for all large } i.$$

A semiscale $\vec{\phi}$ is in a class of relations Γ if both relations

$$U(n,x,y) \iff x \in P \wedge [y \notin P \vee \phi_n(x) \le \phi_n(y)]$$
$$V(n,x,y) \iff x \in P \wedge [y \notin P \vee \phi_n(x) < \phi_n(y)]$$

are in Γ.

It is easy to check that a set P admits a semiscale $\vec{\phi}$ into λ if and only if P is λ-Suslin, *i.e.*, P is the projection of some tree T on $\omega^k \times \lambda$; moreover, T is definable exactly when $\vec{\phi}$ is definable. Sets which admit definable scales are well-behaved in many ways, e.g. we can use a scale on $P \subseteq \mathbb{R}\times\mathbb{R}$ to uniformize P.

Granting projective determinacy, we can prove that every projective set admits a projective scale (Moschovakis); on the other hand it is easy to check that $\{(x,y) : x \text{ is not ordinal definable from } y\}$ does not admit a scale which is OD in a real, granting only that for each y there is some x which is not OD in y. Thus not every "definable" set admits a "definable" scale.

The strongest result we can get with current methods is that inductive sets admit inductive scales, granting inductive determinacy; here P is inductive if P is Σ_1 over the smallest admissible set M which contains the reals, $\mathbb{R} \in M$.

Problem # 2. *Assume* $\mathsf{ZF} + \mathsf{DC} + \mathsf{AD} + \mathbf{V}{=}\mathbf{L}(\mathbb{R})$*; prove or disprove that every coinductive set of reals is* λ*-Suslin for some* λ*.*

Progress report [KMM81A].

The problem was solved by Moschovakis who showed (from $\mathsf{AD}{+}\mathsf{DC}$) that every coinductive pointset admits a scale. If we put

$$\Sigma_0^* = \text{ all Boolean combinations of inductive and coinductive sets}$$

and then define Σ_n^* by counting quantifiers over $\mathbb{R}$ in front of a Σ_0^* matrix in the usual way, then the proof shows that every coinductive set admits a scale $\{\phi_n\}_{n\in\omega}$, where each ϕ_n is a Σ_{n+1}^*-norm, uniformly in n.

Martin and Steel extended the method used by Moschovakis in this proof and showed that

$$\mathsf{ZF} + \mathsf{DC} + \mathsf{AD} + \mathbf{V}{=}\mathbf{L}(\mathbb{R}) \Rightarrow \text{Every } \Sigma_1^2 \text{ set admits a } \Sigma_1^2\text{-scale};$$

this combines with an earlier result of Kechris and Solovay to show that

$$\mathsf{ZF}+\mathsf{DC}+\mathsf{AD}+\mathbf{V} = \mathbf{L}(\mathbb{R}) \Rightarrow \text{A pointset admits a scale if and only if it is } \utilde{\mathbf{\Sigma}}_1^2.$$

Martin then combined these ideas with the technique of the Third Periodicity Theorem (Theorem 6E.1 in [Mos80] or in [Mos09]) and showed that under reasonable hypotheses of determinacy for games on $\mathbb{R}$, (namely, $\mathsf{AD}_{\mathbb{R}}$), the scale property is preserved by the game quantifier $\Game^2$ on $\mathbb{R}$, where

$$(\Game^2\alpha)P(x,\alpha) \iff (\exists\alpha_0)(\forall\alpha_1)(\exists\alpha_2)(\forall\alpha_3)\dots P(x,\langle\alpha_0,\alpha_1,\dots\rangle).$$

This result produces scales for sets that are not $\utilde{\boldsymbol{\Sigma}}^2_1$ in $\mathbf{L}(\mathbb{R})$ and leaves open the general question of the extent of scales in the presence of axioms stronger than AD.

2015 comments.

Moschovakis paper appeared as [Mos83]. The result of Martin and Steel appears in [MS83]. Martin's theorem on preservation of scales under the game quantifier $\Game^2$ is in [Mar83B]. Related results by Steel are in [Ste83B] and [Ste83A]. The later paper introduces the key fine structural analysis of $\mathbf{L}(\mathbb{R})$ via gaps that is now used in the core model induction. These results are reprinted in [CABAL I] and have been further extended by Steel (under appropriate large cardinals or stronger determinacy assumptions), see for instance [Ste08G], [Ste08E], and the introduction [Ste08B]. These extensions are of course needed for core model inductions whose goal is to reach models of strong determinacy assumptions.

#3. The invariance of $\mathbf{L}[T^3]$.

Original problem [KM78A].

Let n be an *odd* integer. Let P be a complete Π^1_n set of reals and assuming PD let $\vec{\phi} = \{\phi_m\}_{m\in\omega}$ be a Π^1_n-scale on P. (It is understood here that each ϕ_m maps P onto an initial segment of the ordinals.) The tree $T^n = T^n(\vec{\phi})$ associated with this scale is defined by

$$T^n = \{\langle\alpha(0),\phi_0(\alpha),\dots,\alpha(k),\phi_k(\alpha)\rangle \,:\, \alpha\in P\}.$$

Let $\mathsf{AD}^{\mathbf{L}(\mathbb{R})}$ be the hypothesis that every set of reals in $\mathbf{L}(\mathbb{R})$ is determined.

PROBLEM # 3. *Assume* $\mathsf{ZF}+\mathsf{DC}+\mathsf{AD}^{\mathbf{L}(\mathbb{R})}$. *Prove or disprove that* $\mathbf{L}[T^3] = \mathbf{L}[T^3(\vec{\phi})]$ *is independent of the choice of the complete* Π^1_3 *set* P *and the particular* Π^1_3*-scale* $\vec{\phi}$ *on* P.

It is known that $\mathbf{L}[T^1] = \mathbf{L}$ (Moschovakis). Also under the above hypothesis it is known that for all odd n and all $T^n = T^n(\vec{\phi})$, $\mathbf{L}[T^n]\cap\mathbb{R} = C_{n+1}$, where C_{n+1} is the largest countable Σ^1_{n+1} set of reals (Harrington-Kechris), so that $\mathbb{R}\cap\mathbf{L}[T^n]$ does not depend on the choice of T^n.

In many ways, the model $\mathbf{L}[T^n]$ is an excellent analog of $\mathbf{L}$ for the $(n+1)$-st level of the analytical hierarchy.

Progress report [KMM81A].

Kechris showed in [Kec81A] that if $T^3 = T^3(\vec{\phi})$ is the tree associated with some Π^1_3-scale $\vec{\phi}$ on a Π^1_3-complete set P and if

$$\widetilde{\mathbf{L}}[T^3] = \bigcup_{\alpha\in\mathbb{R}} \mathbf{L}[T^3, \alpha],$$

then $\mathsf{ZF} + \mathsf{AD} + \mathsf{DC} + \underset{\sim}{\boldsymbol{\delta}}^1_3 \to (\underset{\sim}{\boldsymbol{\delta}}^1_3)^{\underset{\sim}{\boldsymbol{\delta}}^1_3}$ implies that $\widetilde{\mathbf{L}}[T^3]$ is independent of the choice of P and $\vec{\phi}$.

This partial result emphasizes the importance of the question of *the strong partition property* for $\underset{\sim}{\boldsymbol{\delta}}^1_3$ which is still open.

Progress report [KMM83A].

The problem was solved by Becker and Kechris who showed that $L[T^3]$ is independent of the choice of T^3. This is a consequence of the following fact, which is a theorem of $\mathsf{ZF} + \mathsf{DC}$.

THEOREM 1. *Let Γ be an ω-parametrized pointclass closed under $\wedge$ and recursive substitution and containing all recursive sets. Let $P \subset \mathbb{R}$ be a complete Γ set, $\vec{\phi} = \{\phi_i\}_{i\in\omega}$ be an $\exists^{\mathbb{R}}\Gamma$-scale on P such that all norms ϕ_i are regular, and $\kappa = \sup\{\phi_i(x) : i \in \omega, x \in P\}$. Let $T(\vec{\phi})$ be the tree on $\omega \times \kappa$ associated with $\vec{\phi}$. For any set $A \subset \kappa$, if A is $\exists^{\mathbb{R}}\Gamma$-in-the-codes with respect to $\vec{\phi}$ (that is, if the set $\{\langle i, x\rangle \in \omega \times \mathbb{R} : x \in P \wedge \phi_i(x) \in A\}$ is $\exists^{\mathbb{R}}\Gamma$), then $A \in \mathbf{L}[T(\vec{\phi})]$.*

In general, given two such scales $\vec{\phi}$, $\vec{\psi}$, it is not known that $T(\vec{\psi})$ is $\exists^{\mathbb{R}}\Gamma$-in-the-codes with respect to $\vec{\phi}$, so the invariance of $\mathbf{L}[T(\vec{\phi})]$ has not been shown in this generality. However there are special cases where invariance can be proved. Henceforth, assume AD.

In Moschovakis [Mos80, p. 526], a model H_Γ is defined for every pointclass Γ which resembles Π^1_1; this includes the pointclasses Π^1_n for odd n. It follows from Theorem 1, together with known results about the H_Γ's ([Mos80, 8G]), that for any Γ, P, $\vec{\phi}$ such that Γ resembles Π^1_1 and Γ, P, $\vec{\phi}$ satisfy the assumptions of Theorem 1, $L[T(\vec{\phi})] = H_\Gamma$, and hence $\mathbf{L}[T(\vec{\phi})]$ is independent of the choice of P and $\vec{\phi}$. For $\Gamma = \Pi^1_3$ this solves the third problem.

While the invariance problem for $\mathbf{L}[T^n]$ is thus solved for odd n, for even n the situation is still unclear. Call a Σ^1_3-scale on a Π^1_2 set **good** if it satisfies the ordinal quantification property of Kechris-Martin [KM78]. It follows from the above theorem that $\mathbf{L}[T^2]$ is independent of the choice of a complete Π^1_2 set and of the choice of a good scale. Whether or not it is independent of the choice of an arbitrary scale is unknown. For even $n > 2$, it is not known whether there exist any good scales.

2015 comments.

The result by Becker and Kechris appears in [BK84], in the book of proceedings [BMS84].

The problem of the invariance of $L[T^{2n}]$ for $n > 1$ remains open.On the other hand, in [Hjo96C], Hjorth shows that, under $\mathsf{Det}(\underset{\sim}{\boldsymbol{\Pi}}^1_2)$, the model $\mathbf{L}[T^2]$

is independent of the exact choice of T^2. His argument uses forcing to analyze Π^1_3 equivalence relations. In [Hjo95B], he uses properties of $\mathbf{L}[T^2]$ to draw descriptive set theoretic consequences of the assumption that all reals have sharps, in particular showing that if all reals have sharps and MA_{ω_1} holds, then all $\utilde{\mathbf{\Sigma}}^1_3$ sets are Lebesgue measurable.

Meanwhile, developments in inner model theory have provided us both with new methods for analyzing the models $\mathbf{L}[T^n]$, and with the *right* analogues of $\mathbf{L}$ for higher levels of the analytic hierarchy, the fine structural models M_n. Recall that (under appropriate large cardinal assumptions) M_n is the canonical minimal inner model for the assumption that there are n Woodin cardinals. In [Ste95B], (after giving a precise definition of M_n in terms of n-smallness) Steel argues that M_n is Σ^1_n correct, and that $\mathbb{R} \cap M_n = C_n$ for n even, and $\mathbb{R} \cap M_n = Q_n$ for odd n.

 # 4. The strength of $\mathsf{Sep}(\Sigma^1_3)$ in the presence of sharps.
Original problem [KM78A].

Let $(\sharp)$ stand for "$\forall x \subseteq \omega (x^\sharp$ exists)" and let $\mathsf{Sep}(\Sigma^1_3)$ denote "For every $x \subseteq \omega$, every two disjoint $\Sigma^1_3(x)$ sets of reals can be separated by a $\Delta^1_3(x)$ set."

Problem # 4. *Prove or disprove that:*

$$\mathsf{ZFC} + \mathsf{Sep}(\Sigma^1_3) + (\sharp) \Longrightarrow \mathsf{Det}(\utilde{\mathbf{\Delta}}^1_2).$$

Harrington has shown that ZFC+$\mathsf{Sep}(\Sigma^1_3)$ is consistant relative to ZFC. However, using Jensen's Absoluteness Theorem for the core model K (which states that if $(\sharp)$ holds and Σ^1_3 formulas are not absolute for K, then $0^\dagger$ exists) one can see that

$$\mathsf{ZF}+\mathsf{DC}+\mathsf{Sep}(\Sigma^1_3)+(\sharp) \Longrightarrow \forall x \subseteq \omega (x^\dagger \text{ exists }).$$

2015 comments.

Problem # 4 was solved with core model techniques by John Steel, following the ideas of Kechris that are alluded to in the background of the original wording of the problem. The result appears as Corollary 7.14 in [Ste96]. The key lemma is the Σ^1_3 correctness of K if there are no inner models with Woodin cardinals and there exists a measurable cardinal. This is established in [Ste96, Therem 7.9] in the setting of that book, where an additional measurable Ω is assumed in the background and a set sized K is built of height Ω (the measurable whose existence was stated is assumed below Ω); this additional assumption is now known not to be necessary, see [JS13], although this does not affect the argument in Corollary 7.14.

To solve problem # 4 affirmatively, Steel argues that $\mathsf{Sep}(\Sigma^1_3) + (\sharp)$ implies that for every real x there is a proper class model M with $x \in M$, and an ordinal δ such that $V^M_{\delta+1}$ is countable, and δ is Woodin in M. By results of Woodin, this implies $\mathsf{Det}(\utilde{\mathbf{\Delta}}^1_2)$, see [Nee10, Corollary 2.3]. To see that such a model exists, Steel first uses the Σ^1_3-correctness of the Dodd-Jensen core model to deduce that for any real y there exists a proper class model N with $y \in N$

and a measurable cardinal, this is the argument alluded to in the background of the problem (if we want to make do without citing [JS13], then we need to replace the Dodd-Jensen core model with Mitchell's core model, and use Mitchell's Σ^1_3-correctness theorem instead, to ensure that there is such an N with two measurable cardinals).

Once we have N, Steel argues that (by choosing the real y appropriately), the K_x construction inside N must fail (and therefore $(K^c_x)^N$ reaches a Woodin cardinal, and an iterate of an appropriate hull of $(K^c_x)^N$ is the model M as needed). To do this, the argument is by contradiction: If $(K_x)^N$ exists, then it is Σ^1_3-correct inside N. But there is a $\Delta^1_3(x)$ well-ordering of the reals of $(K_x)^N$, which implies the failure of $\mathsf{Sep}(\Sigma^1_3)$ inside $(K_x)^N$. The point is that we can choose y to ensure that $\mathsf{Sep}(\Sigma^1_3)$ relativizes down from V to N, and the correctness of K_x inside N implies that it further relativizes down from N to $(K_x)^N$, which is impossible.

It is still open whether there is a Σ^1_3-correctness theorem for K (in the absence of Woodin cardinals) without additional assumptions beyond the existence of sharps.

 # 5. A classification of functions on the Turing degrees.

Original problem [KM78A].

$\mathcal{D}$ is the set of Turing degrees. A property P of degrees holds almost everywhere (a.e.) iff $\exists \mathbf{c}\, \forall \mathbf{d} \geq \mathbf{c}\, P(\mathbf{d})$. For $f, g \colon \mathcal{D} \to \mathcal{D}$, let $f \leq_{\mathrm{m}} g$ iff $f(\mathbf{d}) \leq g(\mathbf{d})$ a.e. A function $f \colon \mathcal{D} \to \mathcal{D}$ is **representable** iff

$$\exists F \colon {}^\omega\omega \to {}^\omega\omega\, \forall x\, (\deg(F(x)) = f(\deg(x))).$$

PROBLEM # 5. *Working in* $\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}$, *settle the following conjectures of D. Martin:*

(a) If $f \colon \mathcal{D} \to \mathcal{D}$ *is representable and* $\mathbf{d} \not\leq f(\mathbf{d})$ *a.e., then* $\exists \mathbf{c}(f(\mathbf{d}) = \mathbf{c})$ *a.e.*
(b) $\leq_{\mathrm{m}}$ *is a prewellorder of* $\{f : f$ *is representable and* $\mathbf{d} \leq f(\mathbf{d})$ *a.e.*$\}$.

Further, if f *has rank* α *in* $\leq_{\mathrm{m}}$, *then* f' *has rank* $\alpha + 1$, *where* $f'(\mathbf{d}) = f(\mathbf{d})'$, *the Turing jump of* $f(\mathbf{d})$.

REMARKS. With regard to (a), it is well known that if $f(\mathbf{d}) \leq \mathbf{d}$ and $\forall \mathbf{c}\, (\mathbf{c} \leq f(\mathbf{d})$ a.e.), then $f(\mathbf{d}) = \mathbf{d}$ a.e. It is known that conjecture (b) is true when restricted to uniformly representable f so that $\mathbf{d} \leq f(\mathbf{d})$ a.e. (A function f is **uniformly representable** if there is an $F \colon {}^\omega\omega \to {}^\omega\omega$ such that for all x, we have $\deg(F(x)) = f(\deg(x)))$ and, moreover, there is a $t \colon \omega \to \omega$ such that for all x and y, if $x \equiv_T y$ via e then $F(x) \equiv_T F(y)$ via $t(e)$.) It is conjectured that every representable $f \colon \mathcal{D} \to \mathcal{D}$ is uniformly representable.

A proof of conjecture (b) would yield a strong negative answer to a question of G. Sacks: is there a degree invariant solution to Post's problem?

Progress report [KMM81A].

It follows from unpublished results of Kechris and Solovay that $\mathsf{ZF}+\mathsf{AD}+\mathsf{DC}+\mathbf{V}=\mathbf{L}(\mathbb{R})$ implies that every function $f\colon \mathcal{D}\to\mathcal{D}$ on the degrees is representable.

Although this has no direct bearing on a possible solution of the fifth problem, it underscores the generality of the question.

Progress report [KMM83A].

T. Slaman and J. Steel have proved two theorems relevant to Problem # 5. The first verifies a special case of conjecture (a):

THEOREM 2. $(\mathsf{ZF}+\mathsf{AD}+\mathsf{DC})$. *Let* $f\colon\mathcal{D}\to\mathcal{D}$ *be such that* $f(\mathbf{d})<\mathbf{d}$ *a.e.; then for some* c, $f(\mathbf{d})=\mathbf{c}$ *a.e.*

The second verifies a special case of conjecture (b). Call $f\colon\mathcal{D}\to\mathcal{D}$ **order-preserving a.e.** iff $\exists\mathbf{c}\,\forall\mathbf{a},\mathbf{b}\geq\mathbf{c}\,(\mathbf{a}\leq\mathbf{b}\Rightarrow f(\mathbf{a})\leq f(\mathbf{b}))$.

THEOREM 3. $(\mathsf{ZF}+\mathsf{AD}+\mathsf{DC})$. *Let* $f\colon\mathcal{D}\to\mathcal{D}$ *be order-preserving a.e. and such that* $\mathbf{d}<f(\mathbf{d})$ *a.e. Then either*

(i) $\exists\alpha<\omega_1\,(f(\mathbf{d})=\mathbf{d}^\alpha\,a.e.)$, *or*
(ii) For a.e. $\mathbf{d}$, $\forall\alpha<\omega_1^{\mathbf{d}}\,(f(\mathbf{d})>\mathbf{d}^\alpha)$.

(Here ω_1 is the least uncountable ordinal, and ω_1^d is the least d-admissible ordinal greater than ω.)

2015 comments.

Problem # 5 is commonly known as "Martin's Conjecture". The partial results by Slaman-Steel appear in [SS88]. A minor notational change takes place in the paper. First of all, rather than looking at functions $F\colon {}^\omega\omega\to{}^\omega\omega$, Slaman and Steel work with functions $F\colon 2^\omega\to 2^\omega$. Second, rather than working with (representable) functions $f\colon\mathcal{D}\to\mathcal{D}$, they directly work with **degree invariant** functions $f\colon 2^\omega\to 2^\omega$, that is, functions such that $f(x)\equiv_T f(y)$ whenever $x\equiv_T y$.

Their approach is through the combinatorics of pointed trees. A tree $T\subseteq 2^{<\omega}$ is **pruned** iff it has no terminal nodes. It is **perfect** iff it is nonempty, pruned and the set $[T]$ of infinite branches through T is a perfect subset of 2^ω, that is, iff T is nonempty and any node in T has two incompatible extensions in T. We say that T is **pointed** iff T is perfect and for all $x\in[T]$ we have that $T\leq_T x$. The relevance of the notion comes from noting that Martin's argument showing that the cone filter is an ultrafilter on degrees actually gives us that whenever $P\subseteq 2^\omega$ contains a cone of degrees in the sense that for any x there is a $y\geq_T x$ with $y\in P$, then there is a pointed tree T such that $[T]\subseteq P$.

Fixing a standard enumeration of Turing machines with oracles, let $\{e\}^x$ denote the e-th real recursive in x. A Turing equivalence $x\equiv_T y$ is realized via a pair (i,j) of natural numbers iff $\{i\}^x=y$ and $\{j\}^y=x$. A degree invariant function f is **uniformly degree invariant** iff there is a pointed tree T and a function $t\colon\omega\times\omega\to\omega\times\omega$ such that for all $x,y\in[T]$, if $x\equiv_T y$ via (i,j), then $f(x)\equiv_T f(y)$ via $t(i,j)$ (this is the precise meaning of the notion of uniform representation discussed in the remarks following the original statement of the problem).

Working in $\mathsf{AD}+\mathsf{DC}$, in [SS88], Slaman-Steel prove that if f is uniformly invariant and not increasing on a cone, then f is constant on a cone. This proves (a) of problem # 5 under Steel's conjecture that (under $\mathsf{AD}+\mathsf{DC}$) all degree invariant functions are uniformly degree invariant. Part (a) remains open in general. They also prove Theorems 2 and 3. The latter can be improved using results of Woodin [Woo08A], so that part (ii) of the conclusion can be strengthened to $f(\mathbf{d}) > \mathcal{O}^{\mathbf{d}}$.

However, the conjecture remains open. A competing conjecture from the theory of Borel equivalence relations is in conflict with Martin's conjecture: Recall that for Polish spaces X and Y and equivalence relations $\equiv$ and $\equiv'$ on X and Y, respectively, we say that $\equiv$ is **Borel reducible** to $\equiv'$, $\equiv \le_B \equiv'$, iff there is a Borel function $f : X \to Y$ such that for all $x, x' \in X$ we have

$$x \equiv x' \iff f(x) \equiv' f(x').$$

An equivalence relation on X is **Borel** iff it is a Borel subset of $X \times X$, and it is **countable** iff all its equivalence classes are countable. A countable Borel equivalence relation is **universal** iff all countable Borel equivalence relations are Borel reducible to it. Kechris has asked, and it is sometimes stated as a conjecture, whether Turing equivalence is universal.

Slaman and Steel have also shown that *arithmetic* equivalence is universal, but the question remains open for Turing equivalence. A positive answer to Kechris's question would contradict Martin's conjecture, since a reduction of $\equiv_T \sqcup \equiv_T$ to $\equiv_T$ would be such that the range of the reduction on one of the copies of $\equiv_T$ is disjoint from a cone.

Simon Thomas [Tho09B] investigates further implications of Martin's conjecture on the theory of countable Borel equivalence relations. Montalbán, Reimann, and Slaman, have shown (in unpublished work) that Turing equivalence is not *uniformly* universal. Complete references, and a thorough discussion of these matters, including a proof of the Slaman-Steel result on arithmetic equivalence, can be found in Marks-Slaman-Steel [MSS14]. The Montalbán-Reimann-Slaman result was briefly outlined in a talk by Slaman at the 2009 VIG [Sla].

 # 6. The extent of definable scales.
Original problem [KMS88A].

Problem # 6. *Assume* $\boldsymbol{\Pi}^1_1$-AD^{Σ_3}. *Do all* $\Game^{\Sigma_2}\boldsymbol{\Pi}^1_1$ *sets admit* $\mathbf{HOD}(\mathbb{R})$ *scales?*

The terminology is explained in Steel's paper [Ste88]. The strongest result in this direction has been Martin's theorem that for $\lambda < \omega_1$ a limit ordinal, $\boldsymbol{\Pi}^1_1$-AD^{λ} implies all $\Game^{\lambda}\boldsymbol{\Pi}^1_1$ sets admit $\Game^{\lambda}\boldsymbol{\Pi}^1_1$ scales ([Mar08B]). Work of Woodin and Steel had shown that a positive answer to #6 implies that some form of definable determinacy (i.e., $\undertilde{\Pi}^1_1$-AD^{Σ_3}) yields an inner model of $\mathsf{AD}_{\mathbb{R}}$.

Steel obtained a positive answer to #6 in February 1984; his results in this area are described in [Ste88].

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There has been a significant amount of additional work on determinacy of long games and regularity of associated sets. In [Ste08C], Steel extends the work in [Ste88] and shows that (under an appropriate determinacy assumption) the pointclass $\Game^{\omega_1}$(open-analytical) has the scale property. The determinacy assumption, that Steel calls ω_1**-open-projective determinacy**, is the statement that all games $\mathcal{G}(T)$ are determined, for T an ω_1-tree definable over $\mathbf{H}(\omega_1)$ from parameters. Here, T is an ω_1-tree iff $T \subseteq \omega^{<\omega_1}$ and T is closed under initial segments. The game $\mathcal{G}(T)$ is the following (closed) game on ω of length ω_1: For any countable α, at stage α, player I plays an integer m_α and player II replies an integer n_α. Letting $\langle\cdot,\cdot\rangle$ denote a (natural) pairing function, let $f\colon \omega_1 \to \omega$ be the function defined at any α by $f(\alpha) = \langle m_\alpha, n_\alpha\rangle$. Player II wins this run of the game iff $f \in [T]$, the set of length-ω_1 branches through T.

The point of the name, ω_1-open-projective, is that an ω_1-tree is definable over $\mathbf{H}(\omega_1)$ from parameters iff it can be coded by a projective set of reals. Neeman proves in [Nee04] that ω_1-open-projective determinacy follows from a traditional large cardinal assumption, namely, that for every real x there is a countable, $\omega_1 + 1$-iterable (coarse) mouse M with $x \in M$ and $M \models$ ZFC^-+"there is a measurable Woodin cardinal", where ZFC^- denotes ZFC without the power set axiom.

Neeman's monograph [Nee04] discusses what is essentially the state of the art in the theory of determinacy of long games, although a few results of Woodin in the area remain unpublished. Details are discussed in the monograph.

Paul Larson has studied the determinacy of a few long games, with emphasis on combinatorics rather than descriptive set theory. See [Lar05] and [LS08], the latter joint with Saharon Shelah.

7. The Kleene ordinal.
Original problem [KMS88A].

Problem # 7. *Let κ be the least ordinal not the order type of a prewellordering of $\mathbb{R}$ recursive in Kleene's ^3E and a real. Assume $\mathsf{AD}^{\mathbf{L}(\mathbb{R})}$. Is κ the least weakly inaccessible cardinal?*

That the answer is positive is an old conjecture of Moschovakis, who had shown that κ is a regular limit of Suslin cardinals ([Mos70A, Mos78]). Steel showed in [Ste81A] that κ is the least regular limit of Suslin cardinals. Thus the problem amounted to bounding the growth of the Suslin cardinals below

κ. Building on work of Kunen and Martin, Jackson has done this for the first ω Suslin cardinals; this work is described in his long paper [Jac88].

In the fall of 1985, Jackson obtained a positive answer to #7. His new work extends the theory presented in [Jac88]. Because of its length and complexity, as of now no one but Jackson has been through this new work.

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Jackson's result remains unpublished. In [Jac89], written up in 1988, he comments in section 3 (page 80):

> Steel has developed a "fine structure theory" for $\mathbf{L}(\mathbb{R})$ assuming $\mathsf{ZF} + \mathsf{AD}$. This suffices to answer certain questions about $\mathbf{L}(\mathbb{R})$, for example, it gives a complete description of the scale property in $\mathbf{L}(\mathbb{R})$. Other problems, however, such as whether every regular cardinal is measurable seem to require a more detailed understanding of $\mathbf{L}(\mathbb{R})$.
>
> Our results provide such a detailed analysis for an initial segment of the $\mathbf{L}_\alpha(\mathbb{R})$ hierarchy. Exactly how far this enables one to go is not clear, and is the subject of current investigation. However, the author has verified that the theory extends through the Kleene ordinal $\kappa = o({}^3E)$, and in fact, considerably beyond. This analysis is quite involved, however, and has not yet been written up. One consequence is the solution to a problem of Moschovakis, who conjectured in $\mathsf{ZF}+\mathsf{AD}+\mathsf{DC}$ that the Kleene ordinal should be the least inaccessible cardinal (this is the seventh Victoria Delfino problem).

As quoted previously, in our comments to problem # 1, in [Jac10, p. 1755], Jackson indicates that the relevant analysis remained unwritten by 2010, and to the best of our knowledge, no alternative approaches (via the **HOD** analysis or otherwise) have been suggested. Nevertheless, portions of the analysis have appeared. In particular, see [Jac91, Jac92], and [Jac10].

8. Regular cardinals in $\mathbf{L}(\mathbb{R})$.

Original problem [KMS88A].

Problem # 8. *Assume* $\mathsf{AD} + \mathbf{V} = \mathbf{L}(\mathbb{R})$. *Are all regular cardinals below* Θ *measurable?*

Moschovakis and Kechris had shown, in ZFC + $\mathsf{AD}^{\mathbf{L}(\mathbb{R})}$, that if κ is regular (in $\mathbf{V}$, where AC holds!) and $\kappa < \Theta^{\mathbf{L}(\mathbb{R})}$, then $\mathbf{L}(\mathbb{R}) \models \kappa$ is measurable. This led them to conjecture a positive answer to #8. Jackson's detailed analysis of cardinals and measures had verified the conjecture for κ below the sup of the first ω Suslin cardinals (cf. [Jac88]).

The only progress on this problem since its addition to the list is Jackson's new work cited above, which presumably yields a positive answer to #8 for κ below the Kleene ordinal.

2015 comments.

Problem # 8 was solved by John Steel using core model techniques, specifically through the beginning of what we now call the **HOD** analysis. What Steel did was to realize, under the assumption of determinacy, the fragment of the model $\mathbf{HOD}^{\mathbf{L}(\mathbb{R})}$ below Θ as a fine structural mouse, specifically as the direct limit of a system whose objects are certain countable mice and whose commuting maps are appropriate iterations. Analysis of this system allows us to conclude (combinatorial or descriptive set theoretic) properties of its direct limit from (fine structural) properties of the mice, and many different results have been established this way. In particular, in [Ste95A] Steel shows:

THEOREM 4. *Assume* $\mathsf{AD}^{\mathbf{L}(\mathbb{R})}$ *and work in* $\mathbf{L}(\mathbb{R})$*. Then for every* $x \in \mathbb{R}$ *and* $\kappa < \underset{\sim}{\boldsymbol{\delta}}^2_1$ *such that* κ *is regular in* $\mathbf{HOD}(x)$*, the following implication holds:*

$$\mathrm{cf}(\kappa) > \omega \textit{ implies } \mathbf{HOD}(x) \models \text{“}\kappa \textit{ is measurable”}.$$

The point is that the order zero measures on κ in $\mathbf{HOD}(x)$ for different x can be "put together" via the directed system that guides the iterations mentioned above. Details can be found in Steel's Handbook article, see [Ste10B, 8.27 Theorem]. The result now follows via reflection.

This analysis of $\mathbf{V}_\Theta \cap \mathbf{HOD}^{\mathbf{L}(\mathbb{R})}$ has been extended by Woodin to a full analysis of **HOD** via a longer directed system, while identifying the correct *hybrid* rather than purely fine structural mice that make up **HOD**. See for instance [SW14] and [Ste10B, 8.25 Lemma]. A similar analysis of the **HOD** of larger models than $\mathbf{L}(\mathbb{R})$ has positioned itself as a key tool in recent work in determinacy (in particular, in the proofs of partial versions of the *mouse set conjecture*), with Grigor Sargsyan being the main force behind these new developments, see [Sar13B].

To illustrate the reach of the **HOD** analysis, we mention some further applications (the list is not exhaustive): Recall that, assuming determinacy, κ^1_{2n+1} is the cardinal predecessor of the projective ordinal $\underset{\sim}{\boldsymbol{\delta}}^1_{2n+1}$. In [Sar13C], Sargsyan proves Woodin's theorem that, under $\mathsf{AD} + \mathbf{V} = \mathbf{L}(\mathbb{R})$, for all $n \in \omega$, κ^1_{2n+3} is the least cardinal δ of **HOD** such that $M_{2n}(\mathbf{HOD}|\delta) \models$ "δ is Woodin." This identifies a purely descriptive set theoretic characterization of cardinals with a fine structural characterization, and provides us with precise information of how much large cardinal strength the relevant cardinals retain when passing from **V** to nice inner models. In [Sar14], Sargsyan uses the **HOD** analysis to prove the strong partition property of $\underset{\sim}{\boldsymbol{\delta}}^2_1$, a result first established in [KKMW81]. In [Nee07B], Neeman uses the analysis to provide a characterization of supercompactness measures for ω_1 in $\mathbf{L}(\mathbb{R})$. In [JKSW13], the authors use the analysis to prove Woodin's result that, under $\mathsf{AD} + \mathbf{V} = \mathbf{L}(\mathbb{R})$, every uncountable cardinal below Θ is Jónsson and, if its cofinality is ω, then it is even Rowbottom. This drastically extend previous results of Kleinberg [Kle77].

 # 9. Large cardinals implying determinacy.

Original problem [KMS88A].

PROBLEM # 9. *Does the existence of a nontrivial, elementary* $j\colon \mathbf{V}_{\lambda+1} \to \mathbf{V}_{\lambda+1}$ *imply* $\utilde{\mathbf{\Pi}}^1_3$ *determinacy?*

The world view embodied in the statements of this and the succeeding problem was seriously mistaken. That view was inspired by Martin's result ([Mar80]) that the existence of a nontrivial, Σ_1-elementary $j\colon \mathbf{V}_{\lambda+1} \to \mathbf{V}_{\lambda+1}$ implies $\utilde{\mathbf{\Pi}}^1_2$ determinacy, together with work of Mitchell [Mit79] which promised to lead to a proof that nothing much weaker than the existence of such an embedding would imply $\utilde{\mathbf{\Pi}}^1_2$ determinacy. Martin naturally conjectured that a nontrivial, fully elementary $j\colon \mathbf{V}_{\lambda+1} \to \mathbf{V}_{\lambda+1}$ would yield PD; hence the inclusion of # 9 on our list.

Partly because this view was so mistaken, progress in this area since 1984 has been dramatic. In February–April of 1984 Woodin showed that the existence of a nontrivial, elementary $j\colon \mathbf{L}(\mathbf{V}_{\lambda+1}) \to \mathbf{L}(\mathbf{V}_{\lambda+1})$ implies PD and in fact $\mathsf{AD}^{\mathbf{L}(\mathbb{R})}$. This was still consistent with the view underlying # 9, and in spirit was a positive answer, although even for $\utilde{\mathbf{\Pi}}^1_3$ determinacy Woodin's result required a hypothesis slightly stronger than allowed in # 9. However, at about the same time Foreman, Magidor and Shelah ([FMS88]) developed a powerful new technique for producing generic elementary embeddings under relatively "weak" large cardinal hypotheses such as the existence of supercompact cardinals. Woodin realized at once the potential in their technique and used it to show, in May 1984, that the existence of a supercompact cardinal implies all projective sets of reals are Lebesgue measurable. Immediately thereafter, Shelah and Woodin improved this to include all sets in $\mathbf{L}(\mathbb{R})$.

If the relationship between large cardinals and determinacy were to exhibit anything like the pattern it had previously, supercompact cardinals had to imply $\mathsf{AD}^{\mathbf{L}(\mathbb{R})}$. In September 1985, Martin and Steel showed that in fact they do (thereby answering # 9 positively). (Their proof of PD is self-contained. Their proof of $\mathsf{AD}^{\mathbf{L}(\mathbb{R})}$ requires work done by Woodin using the generic embedding techniques.) The Martin-Steel theorem required much less than supercompactness; e.g., for $\utilde{\mathbf{\Pi}}^1_{n+1}$ determinacy it required the existence of n "Woodin cardinals" with a measurable above them all. [The notion of a "Woodin cardinal" had been isolated by Woodin in his work on generic embeddings; it is a refinement of a notion due to Shelah.] In May–July of 1986, Martin and Steel pushed the theory of inner models for large cardinals far enough to show that the hypothesis of their theorem was best possible: the existence of n Woodin cardinals does not imply $\utilde{\mathbf{\Pi}}^1_{n+1}$ determinacy. More recently, Woodin has obtained relative consistency results in this direction by a different method; cf. # 10 below.

Unfortunately, with the exception of [FMS88], none of this recent work has been published.

2015 comments.

The relation between determinacy and large cardinals is now well documented. Nonetheless, since it is of fundamental importance to the field and the Cabal, we provide more details here than otherwise strictly necessary.

1. Suppose that δ is an infinite ordinal and that $A \subseteq \mathbf{V}_\delta$. A cardinal $\lambda < \delta$ is **$< \delta$-A-strong** iff for any $\mu < \delta$ there is a nontrivial elementary embedding $j \colon \mathbf{V} \to M$ with critical point λ and such that $j(\lambda) > \mu$, $\mathbf{V}_\mu \subset M$, and $j(A) \cap \mathbf{V}_\mu = A \cap \mathbf{V}_\mu$. The ordinal δ is a **Woodin cardinal** iff it is an inaccessible cardinal and for all $A \subseteq \mathbf{V}_\delta$ there is a $< \delta$-A-strong cardinal.

The Shelah-Woodin results on Lebesgue measurability of all sets of reals in $\mathbf{L}(\mathbb{R})$ in the presence of large cardinals appear in [SW90]. The paper defines the notions now known as Shelah and Woodin cardinals, although the notation it uses is different, see page 384, Definition 3.5 in page 387, and Definition 4.1 in page 392. In page 384 we read:

> We define here two large cardinals: $\mathrm{Pr}_a(\lambda, f)$, $\mathrm{Pr}_a(\lambda)$ by Shelah (Definition 3.5) and $\mathrm{Pr}_b(\lambda)$ by Woodin – now called a Woodin cardinal.

A cardinal λ is **Shelah** (in the paper, $\mathrm{Pr}_a(\lambda)$ *holds*) iff for every $f \colon \lambda \to \lambda$ there is a nontrivial elementary embedding $j \colon \mathbf{V} \to M$ with critical point λ and such that $\mathbf{H}(j(f)(\lambda)) \subset M$ and $M^{<\lambda} \subset M$.

Woodin cardinals (cardinals δ for which $\mathrm{Pr}_b(\delta)$ holds, in the terminology of the paper) are defined as those cardinals δ such that for any $f \colon \delta \to \delta$ there is a $\lambda < \delta$ with $f``\lambda \subseteq \lambda$ and such that there is an elementary embedding $j \colon \mathbf{V} \to M$ with critical point λ and such that $\mathbf{H}(j(f)(\lambda)) \subset M$. This is easily seen to be equivalent to the definition stated above.

Woodinness was instantly recognized as a pivotal large cardinal notion, and its properties were immediately studied in detail. It plays a key role in modern proofs of determinacy. The Martin-Steel results appear in [MS88] and [MS89], which also mark the first appearance of the term *Woodin cardinal* in the literature, with [MS88] having been published a couple of years before [SW90].

2. There are two special examples of generic embeddings associated with Woodin cardinals, both due to Woodin: Those coming from (one of the versions of) the *stationary tower* forcing, and those coming from the *extender algebra*. The stationary tower appears first in [Woo88]. The monograph [Lar04] by Paul Larson presents the forcing and its properties in detail. For the extender algebra, see for example [Ste10B], or [Nee04, § 4B]. In typical proofs of determinacy, generic iterations are arranged so that a model is obtained whose reals are the reals of $\mathbf{V}$, but the iteration ensures they are suitably generic. To see how determinacy can follow from this, see for instance [Nee95].

We actually have reasonable flexibility on the details of how to carry out the iteration. The following quote is from page 1880 of Neeman's Handbook article [Nee10]:

> $\mathsf{AD}^{\mathbf{L}(\mathbb{R})}$ from infinitely many Woodin cardinals and a measurable cardinal above them is due to Woodin, proved using the methods of stationary tower forcing and an appeal to the main theorem, Theorem 5.11, in Martin-Steel [MS89]. A proof using Woodin?s genericity iterations and fine structure instead of stationary tower forcing is due to Steel, and the proof reached in this chapter (using a second form of genericity iterations and no fine structure) is due to Neeman.

3. The importance of Woodin cardinals was further confirmed once it was recognized that understanding their properties is a key step in the development of the inner model program. Basic to the theory of fine structural models is the way that mice can be compared. These comparisons were linear for the cardinals that current technology could reach, and that imposed serious limitations on the nature of the corresponding models. For instance, all of them admitted $\utilde{\mathbf{\Delta}}^1_3$ well-orderings of their set of reals. This of course is impossible if all projective sets are measurable, which meant that if inner model theory had any hope of reaching supercompact cardinals, essential changes were needed. Increasing the complexity of the comparison process (rather than linear, now having a tree-like structure, what we now call *iteration trees*) was one of these changes. The development of the appropriate fine structure followed shortly thereafter. See [MaS94] and [MiS94].

In fact, the effect of Woodin cardinals could be measured quite precisely, in the complexity of the reals present in canonical inner models, in the amount of determinacy outright provable or provably consistent, in the amount of correctness that a model would satisfy or that could be forced of an iterate of the model. The end result is that the set theoretic landscape transformed significantly thanks to their introduction.

4. For proofs of determinacy from large cardinals, see [Nee10]. As mentioned in the quote above, the result is that from a technical weakening of the assumption that there are ω Woodin cardinals and a measurable above them all we can prove that determinacy holds in $\mathbf{L}(\mathbb{R})$. In fact, the existence of a sufficiently iterable version of $M_\omega^\sharp$ suffices. The proof goes by deriving determinacy of a pointclass from the existence of homogeneously Suslin representations for the sets in the class. Using this one can show that if there are n Woodin cardinals with a measurable above, then all $\utilde{\mathbf{\Pi}}^1_{n+1}$ sets of reals are determined. The optimal result (see [Nee95])) is that if for every real x there is a suitable model M that is iterable and contains x and n Woodin cardinals, then $\mathsf{Det}(\utilde{\mathbf{\Pi}}^1_{n+1})$ holds. The case $n = 0$, where the assumption reduces to asserting that $x^\sharp$ exists for all reals x, is a result of Martin [Mar70A], and from there the proof proceeds by induction: Given a $\utilde{\mathbf{\Pi}}^1_{n+1}$ definition of a set, the quantifiers over reals are turned into quantifiers over iteration trees and their branches, and these are then used to obtain the relevant homogeneously Suslin representations. The arguments can be pushed much further, and the determinacy of stronger pointclasses that $\mathcal{P}(\mathbb{R}) \cap \mathbf{L}(\mathbb{R})$ is provable by similar methods from large cardinals still in the region of Woodin cardinals. In particular, well before reaching the level of rank-to-rank embeddings or even supercompactness.

That assuming fewer than n Woodin cardinals does not suffice follows from inner model theory, since from the existence of n Woodin cardinals, a fine structural model with n Woodin cardinals can be obtained, in which the reals admit a Δ^1_{n+2} well-ordering ([MaS94, Ste95C]), and therefore $\mathsf{Det}(\Pi^1_{n+1})$ fails in the model by [Kan03, Exercise 27.14]. The point is that Kechris and Martin showed that $\mathsf{Det}(\Pi^1_{n+1})$ implies that Σ^1_{n+2} sets are Lebesgue measurable, but no well-ordering of the reals can be measurable, by a result that goes back to Sierpiński [Sie20].

5. Half of Woodin's **derived model theorem** gives us that ω Woodin cardinals suffice to establish the *consistency* of determinacy in $\mathbf{L}(\mathbb{R})$, and appropriate weakenings hold if only finitely many Woodin cardinals are present. Specifically, as shown for instance in [Nee95, Corollary 2.3], if δ is Woodin and G is $\mathrm{Col}(\omega, \delta)$-generic over $\mathbf{V}$, then $\mathsf{Det}(\Delta^1_2)$ holds in $\mathbf{V}[G]$ (and therefore also $\mathsf{Det}(\Sigma^1_2)$, by a result of Martin, see [KW10, Theorem 6.3].) Also:

Theorem 5. *If λ is a limit of Woodin cardinals, G is $\mathrm{Col}(\omega, < \lambda)$-generic over $\mathbf{V}$, and $\mathbb{R}^* = \bigcup_{\alpha<\lambda} \mathbb{R} \cap V[G|\alpha]$, then $\mathbf{L}(\mathbb{R}^*)$ is a model of determinacy.*

In fact, we have that $\mathbb{R}^* = \mathbb{R} \cap \mathbf{V}(\mathbb{R}^*)$ and, letting Γ denote the collection of all sets of reals $A \subseteq \mathbb{R}^*$ in $\mathbf{V}(\mathbb{R}^*)$ such that $\mathbf{L}(A, \mathbb{R}^*) \models \mathsf{AD}^+$, then we have that $\mathbf{L}(\Gamma, \mathbb{R}^*)$ is also a model of determinacy.

This is the *derived model construction*; here, AD^+ denotes Woodin's strengthening of the axiom of determinacy, see problem # 14 below.

The other half of the theorem recovers models of choice with Woodin cardinals from models of determinacy. In particular, if $\mathsf{AD}^{\mathbf{L}(\mathbb{R})}$ holds, then in a forcing extension there is a model of choice with ω Woodin cardinals. The argument proceeds by identifying (in the ground model) for all n, models with n Woodin cardinals that can be extended to larger models with $n+1$ Woodin cardinals. A variant of Prikry forcing is then used to dovetail these models together into one with ω Woodin cardinals.

For proof of these results and of particular versions, see [Ste09] (where models of determinacy are obtained from large cardinals), [KW10] (where Woodin cardinals are obtained in **HOD** and in **HOD**-like models under the assumption of determinacy), [ST10, Zhu2012, available at `http://scholarbank.nus.edu.sg/handle/10635/34464`], and [Zhu10], among others.

In particular, in [KW10, Theorem 6.1] it is shown that $\omega_2^{\mathbf{L}[x]}$ is a Woodin cardinal in $\mathbf{HOD}^{\mathbf{L}[x]}$ for a Turing cone of reals x, under the assumption of $\mathsf{DC} + \mathsf{Det}(\Delta^1_2)$. Also ([KW10, Theorem 5.1]):

Theorem 6. *If $\mathsf{AD}^{\mathbf{L}(\mathbb{R})}$ holds, then $\Theta^{\mathbf{L}(\mathbb{R})}$ is a Woodin cardinal in $\mathbf{HOD}^{\mathbf{L}(\mathbb{R})}$. In fact, if $\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}$ holds, X, Y are sets of reals, and $\Theta_{X,Y}$ denotes the* *supremum of the lengths of those pre-well-orderings of subsets of $\mathbb{R}$ that are ordinal definable from X, Y, then $\mathbf{HOD}_X$ is a model of* ZFC *where $\Theta_{X,Y}$ is Woodin.*

It is thanks to the generality of the second clause in the theorem (since X, Y are arbitrary) that we can obtain models with stacks of Woodin cardinals.

6. Martin [Mar70A] showed that if all reals have sharps, then $\mathsf{Det}(\boldsymbol{\Pi}^1_1)$ holds. Conversely, Harrington [Har78] showed that if x is a real and $\mathsf{Det}(\Pi^1_1(x))$ holds, then $x^\sharp$ exists.

Similarly, from the existence of n Woodin cardinals we can by forcing obtain models of $\mathsf{Det}(\boldsymbol{\Pi}^1_{n+1})$ (see [Nee95]), and this is optimal in the sense that from this much determinacy we can find inner models with n Woodin cardinals and, in fact, for all reals x we can find a suitably iterable (coarse) model containing x and n Woodin cardinals. This was claimed by Woodin in 1995. For $n = 1$, it can be deduced from the arguments in [KW10]. In fact, we get that if both pointclasses $\boldsymbol{\Pi}^1_1$ and Π^1_2 are determined, then $M_1^\sharp$ exists and is ω_1-iterable. Surprisingly, the general argument (for $n > 1$) remains unpublished. However, in the preprint [SUW14], Schindler, Uhlenbrock, and Woodin prove the following theorem:

Theorem 7. *Assume $\mathsf{Det}(\Pi^1_{n+1})$ and $\mathsf{Det}(\boldsymbol{\Pi}^1_n)$. If there is no projective sequence of length ω_1 of distinct reals, then $M_n^\sharp$ exists and is ω_1-iterable.*

The argument uses fine structure, and can be adapted to show:

Theorem 8. *If $\mathsf{Det}(\boldsymbol{\Pi}^1_{n+1})$ holds, then $M_n^\sharp(x)$ exists and is ω_1-iterable for all reals x.*

On the other hand, it appears to be open whether, without additional assumptions, for $n > 1$ (or even just for odd $n > 1$) the determinacy of both pointclasses $\boldsymbol{\Pi}^1_n$ and Π^1_{n+1} implies the existence and ω_1-iterability of $M_n^\sharp$. (In personal communication, Yizheng Zhu has suggested an approach that would solve the problem affirmatively for $n = 3$ and, appropriately generalized, for all odd n.)

7. That Θ is Woodin in $\mathbf{HOD}$ if AD holds and $\mathbf{V} = \mathbf{L}(\mathbb{R})$ has been significantly generalized and is part of the $\mathbf{HOD}$ analysis mentioned in connection with problem # 8. Solovay introduced in [Sol78B] the sequence $\langle \vartheta_\alpha \mid \alpha \leq \Omega \rangle$ as a way of measuring the strength of determinacy models. Assume determinacy. Recall that ϑ_0 is the supremum of all ordinals α for which there is an ordinal definable pre-well-ordering of a subset of $\mathbb{R}$ of length α. If ϑ_α is defined for all $\alpha < \beta$, and β is limit, then ϑ_β is defined as their supremum. Finally, if ϑ_α is defined and is less than Θ, then $\vartheta_{\alpha+1}$ is the supremum of the lengths of all pre-well-orderings of subsets of $\mathbb{R}$ that are definable from ordinals and a set of reals of Wadge rank ϑ_α. The sequence ends once an ordinal Ω is reached such that $\vartheta_\Omega = \Theta$. In $\mathbf{L}(\mathbb{R})$, $\Theta = \vartheta_0$, but longer sequences are possible and correspond to models of stronger versions of determinacy. It turns out that all Θ_α with

α successor are Woodin cardinals in **HOD**. The situation at limit ordinals is more delicate and still being explored, see [Sar15]. Conversely, starting with models with many Woodin cardinals, the derived model construction provides us with models of strong versions of determinacy, see for instance [Ste08A].

8. Although no longer relevant to the goal of deriving determinacy from large cardinals, Woodin's original approach led to the development of the theory of large cardinals past the level of rank-to-rank embeddings. The motivation was the realization that there was a strong analogy between the theory of $\mathbf{L}(\mathbb{R})$ in the presence of determinacy, and the theory of $\mathbf{L}(\mathbf{V}_{\lambda+1})$ in the presence of nontrivial embeddings $j\colon \mathbf{L}(\mathbf{V}_{\lambda+1}) \to \mathbf{L}(\mathbf{V}_{\lambda+1})$ with λ being the supremum of the associated critical sequence. Some results illustrating this can be found in [Kaf04], where versions of the coding lemma are established. For more of the theory of very large cardinals, see also [Lav01, Dim11], and Woodin's work on suitable extender models, [Woo10B, Woo11].

10. Supercompacts in $\mathbf{HOD}^{\mathbf{L}(\mathbb{R})}$.
Original problem [KMS88A].

PROBLEM # 10. *Assume* $\mathsf{AD}^{\mathbf{L}(\mathbb{R})}$. *Does* $\mathbf{HOD}^{\mathbf{L}(\mathbb{R})}$ *satisfy "*$\exists\kappa\,(\kappa$ *is* 2^κ*-supercompact)"?*

Becker and Moschovakis ([BM81]) had shown that $\mathbf{HOD}^{\mathbf{L}(\mathbb{R})} \models \exists\kappa(O(\kappa) = \kappa^+)$. Martin (unpublished) then showed $\mathbf{HOD}^{\mathbf{L}(\mathbb{R})} \models \exists\kappa(\kappa$ is μ-measurable). Steel (unpublished) then showed $\mathbf{HOD}^{\mathbf{L}(\mathbb{R})} \models \exists\kappa(\kappa$ is λ-strong, where $\lambda > \kappa$ is measurable). Inspired by these results, the Cabal conjectured that the model $\mathbf{HOD}^{\mathbf{L}(\mathbb{R})}$ satisfies all large cardinal hypotheses weaker than that which implies $\mathsf{AD}^{\mathbf{L}(\mathbb{R})}$ (which is false in $\mathbf{HOD}^{\mathbf{L}(\mathbb{R})}$). Problem #10 resulted from our mistaken guess as to what these hypotheses are.

The Woodin-Shelah theorem that the existence of supercompacts implies all sets in $\mathbf{L}(\mathbb{R})$ are Lebesgue measurable settles #10 negatively, since, assuming $\mathsf{AD}^{\mathbf{L}(\mathbb{R})}$, $\mathbf{HOD}^{\mathbf{L}(\mathbb{R})} \models$ "There is a wellorder of $\mathbb{R}$ in $\mathbf{L}(\mathbb{R})$". However, except for the mistake about the cardinals involved, the answer to #10 is positive. Woodin has recently (February 1987) shown that, assuming $\mathsf{AD}^{\mathbf{L}(\mathbb{R})}$, $\mathbf{HOD}^{\mathbf{L}(\mathbb{R})} \models \exists\kappa(\kappa$ is a Woodin cardinal), and under the same assumption found a natural submodel of $\mathbf{HOD}^{\mathbf{L}(\mathbb{R})}$ satisfying "There are ω Woodin cardinals". The work of Martin, Steel and Woodin referred to in the discussion of #9, together with further work of Woodin reducing its large cardinal hypothesis, shows that $\mathsf{AD}^{\mathbf{L}(\mathbb{R})}$ follows from the existence of ω Woodin cardinals with a measurable above them all, so that Woodin's recent work is in spirit a positive answer to #10.

2015 comments.

The remarks we gave on problem # 9 apply here as well. The paper [KW10] shows how to find Woodin cardinals in **HOD**. Assuming strong forms of determinacy, the question of precisely which large cardinals can be present in

HOD remains open, with modern research in descriptive inner model theory ([Sar13B]) motivated by the expectation that at least a very large initial segment of the large cardinal hierarchy should be realized within the **HOD** models of strong models of determinacy.

11. The GCH in $\mathbf{HOD}^{\mathbf{L}(\mathbb{R})}$. Original problem [KMS88A].

PROBLEM # 11. *Assume* $\mathsf{AD}^{\mathbf{L}(\mathbb{R})}$. *Does* $\mathbf{HOD}^{\mathbf{L}(\mathbb{R})}$ *satisfy the* GCH*?*

Becker ([Bec80]) has shown that, assuming $\mathsf{AD}^{\mathbf{L}(\mathbb{R})}$, $\mathbf{HOD}^{\mathbf{L}(\mathbb{R})} \models 2^\kappa = \kappa^+$ for many cardinals κ.

There has been little progress on this question since January 1984. Woodin's recent work on large cardinals in $\mathbf{HOD}^{\mathbf{L}(\mathbb{R})}$ does show that, letting $\delta = (\underset{\sim}{\delta}^2_1)^{\mathbf{L}(\mathbb{R})}$ and $\Theta = \Theta^{\mathbf{L}(\mathbb{R})}$, $\mathbf{HOD}^{\mathbf{L}(\mathbb{R})} \models \delta$ is Θ-strong. It follows by an easy reflection argument that if $\mathbf{HOD}^{\mathbf{L}(\mathbb{R})}$ satisfies the GCH below $(\underset{\sim}{\delta}^2_1)^{\mathbf{L}(\mathbb{R})}$, then it satisfies the GCH.

2015 comments.

Steel's analysis of **HOD** below $\underset{\sim}{\delta}^2_1$, mentioned in the solution to problem #8, also solves #11, see [Ste10B, 8.22 Corollary]. Beyond the fine structural analysis, Steel's argument uses the result mentioned in the original wording of the problem, that under AD, $\underset{\sim}{\delta}^2_1$ is strong up to Θ in $\mathbf{HOD}^{L(\mathbb{R})}$, see [KW10]. It also uses that there is a subset P of Θ in $L(\mathbb{R})$ such that $\mathbf{HOD}^{L(\mathbb{R})} = L(P)$. Both these results are due to Woodin. The second follows from the analysis of the Vopěnka algebra, or see [SW14].

The argument generalizes to the **HOD** of larger models of determinacy, as long as the models allow a version of the **HOD** analysis. At the moment, this falls within the region below a Woodin limit of Woodin cardinals or, in terms of determinacy assumptions, up to $\mathsf{AD}_\mathbb{R}$+"Θ is regular" ([Sar15, Sar13A]), but the expectation is that the result should hold in general.

12. Projective uniformization, measure, and category. Original problem [KMS88A].

PROBLEM # 12. *Does the theory* ZFC+ *"Every projective relation can be uniformized by a projective function" + "Every projective set is Lebegue measurable and has the property of Baire" prove* PD*?*

Woodin ([Woo82]) showed that the theory in question proves $\forall x \subseteq \omega (x^\dagger$ exists) and more in this direction, together with some other consequences of PD, and conjectured a positive answer to #12.

There has been no direct progress on this problem since 1984.

2015 comments.

Although the expectation was a positive answer, Problem # 12 was solved negatively by Steel in 1997. The precise strength of the theory in question is that of ZFC together with the existence of a cardinal δ such that there is an ω sequence cofinal in δ of cardinals that are δ-strong. Recall that $\kappa < \delta$ is **δ-strong** iff for all $X \subseteq \delta$ there is an elementary embedding $j\colon \mathbf{V} \to M$ with critical point κ and such that $X \in M$.

Details can be found as handwritten notes by Schindler [Sch], and in Philipp Doebler's diploma thesis [Doe06]. Steel showed that the large cardinal mentioned above suffices to produce a model of the theory under consideration, and Schindler proved that this is indeed an equiconsistency, see [Sch02, Theorem 9.1].

Steel's argues by forcing with $\mathrm{Col}(\omega, \delta)$. In the resulting model, all projective sets are Lebesgue measurable and have the Baire property. Now, if there is a cardinal δ as required, then there is a minimal, fully iterable, fine structural inner model witnessing that there is such a cardinal δ. Call it $L[E]$ and work in this model. We then have that $V[G]$ also satisfies projective uniformization. Finally, $L[E]$ is Σ^1_3 elementary in $L[E][G]$, because $L[E]$ is the core model of any of its forcing extensions. But $L[E]$ admits a $\utilde{\mathbf{\Sigma}}^1_3$ well-ordering of its reals, which implies that there is a non-determined $\utilde{\mathbf{\Delta}}^1_2$ set in K since, by the same argument of [KW10, Theorem 6.3], the assumption of $\mathsf{Det}(\utilde{\mathbf{\Delta}}^1_2)$ implies $\mathsf{Det}(\utilde{\mathbf{\Pi}}^1_2)$. But, as explained in *4.* of the comments to problem # 9, $\mathsf{Det}(\utilde{\mathbf{\Pi}}^1_2)$ gives us the Lebesgue measurability of all $\utilde{\mathbf{\Sigma}}^1_3$ sets, but no well-ordering of the reals is measurable. One then computes that the non-determinacy of a $\utilde{\mathbf{\Delta}}^1_2$ set is a $\utilde{\mathbf{\Pi}}^1_3$ statement, and the elementarily of $L[E]$ in $L[E][G]$ then gives us that already $\mathsf{Det}(\utilde{\mathbf{\Delta}}^1_2)$ fails in $L[E][G]$.

The proof of the converse, that the theory of problem # 12 gives us an inner model with a cardinal δ and an ω-sequence of cardinals cofinal in δ and δ-strong, is due to Hauser and Schindler, and appears in [HS00], using results of Schindler on the complexity of $K \cap \mathbf{H}(\omega_1)$, see [HS00, Theorems 3.4 and 3.6]. The paper uses the additional assumption that $\mathbb{R}^\sharp$ exists in order to implement the core model theory of [Ste96]. What Schindler does in [Sch02] is to show that, at the level of the theories under consideration, core model theory works without this additional assumption and therefore provides us with a genuine equiconsistency. Naturally, we could now invoke [JS13] instead to achieve the same effect.

From further results in [HS00] and the same argument from [Sch02], we also have that the ZF version of the theory from problem # 12 (without the additional assumption of the axiom of choice) gives us an inner model with a cardinal δ and an ω-sequence of cardinals cofinal in δ and strong up to δ, that is, λ-strong for all $\lambda < \delta$. This is also an equiconsistency, as can be verified by starting with the corresponding minimal $L[E]$ model for this large cardinal assumption, and forcing now with the *symmetric* collapse of the supremum of the strong cardinals.

There are at least two subtler versions of problem # 12 that remain open. In the first, we strengthen the theory by changing the assumption of projective uniformization with its level-by-level version, namely, that for each n, any $\underset{\sim}{\mathbf{\Pi}}^1_{2n+1}$ subset of $\mathbb{R}^2$ can be uniformized by a function with a $\underset{\sim}{\mathbf{\Pi}}^1_{2n+1}$ graph. Steel has shown that this version implies $\mathsf{Det}(\underset{\sim}{\mathbf{\Delta}}^1_2)$, combining the solution of problem # 4 with Woodin's work in [Woo82], see [Ste96, Corollary 7.14].

In the second, we replace the assumption with its lightface version, that is, that all lightface projective subsets of $\mathbb{R}^2$ can be uniformized by a function with a lightface projective graph.

13. The cofinal branches hypothesis (CBH).

The **cofinal branches hypothesis**, introduced by Martin and Steel [MaS94, pp. 50–53], is the statement that every countable iteration tree on $\mathbf{V}$ has at least one cofinal well-founded branch.

PROBLEM # 13. *Does* CBH *hold?*

The **unique branches hypothesis**, UBH, also introduced by Martin-Steel [MaS94], is the statement that every countable iteration tree on $\mathbf{V}$ has at most one cofinal well-founded branch. As long as the iteration tree $\mathcal{T}$ under consideration is sufficiently closed, UBH for $\mathcal{T}$ implies CBH for $\mathcal{T}$.

2015 comments. About two years after being formally stated, Woodin refuted UBH using large cardinals at the level of embeddings $j : V_\lambda \to V_\lambda$. Later, in 1999, he also refuted CBH, from the existence of a supercompact with a Woodin above, showing from these assumptions that there is an iteration tree of length ω^2 with no cofinal well-founded branch. The tree is formed by an ultrapower by an extender, followed by an ω sequence of alternating chains on the ultrapower model.

The argument also refutes UBH from the same assumptions, the counterexample being a single ultrapower, now followed by an alternating chain on the ultrapower model, both of whose branches are well-founded.

Details for the case of UBH were presented by Woodin at a meeting on Core Model Theory at AIM, the American Institute of Mathematics, in December 2004. Later, Neeman and Steel significantly lowered the large cardinal assumption needed for both results, to something weaker than the existence of a cardinal strong past a Woodin. More precisely, Neeman and Steel obtained their counterexamples (using the same tree structure as in Woodin's results) from the assumption that there exists a cardinal δ and an extender F such that (1) F has critical point below δ, support δ, and is δ-strong, and (2) δ is Woodin in the smallest admissible set containing $V_\delta \cup \{F\}$.

Details, including a discussion of revised versions of both hypotheses that remain open, together with partial positive results, can be found in [NS06].

14. ∞-Borel sets.

The last problem in the list is a version of the AD^+ question of whether Woodin's AD^+ is equivalent to AD. See [CK11, Section 2], and the references suggested there, for an introduction to AD^+.

In the context of ZF, AD^+ is the conjunction of:

- $\mathsf{DC}_{\mathbb{R}}$.

  Recall that $\mathsf{DC}_{\mathbb{R}}$, or (perhaps more precisely) $\mathsf{DC}_\omega(\mathbb{R})$, is the statement that whenever $R \subseteq \mathbb{R}^2$ satisfies that for any real x there is a y with $x\ R\ y$, then there is a function $f : \omega \to \mathbb{R}$ such that $f(n)\ R\ f(n+1)$ for all n. Equivalently, any tree T on a subset of $\mathbb{R}$ with no end nodes has an infinite branch.
- All sets of reals are ∞-Borel.

  Informally, a set is **∞-Borel** iff it can be generated from open sets by closing under the operations of complementation and well-ordered union. Since we are in a choiceless context, it is better to define that a set is ∞-Borel iff it is the interpretation A_T of an ∞-Borel code T. The set BC of **Borel codes** can in turn be defined recursively in several (rather different) ways. For the sake of concreteness, we specify that $T \in \mathsf{BC}$ iff one of the following holds:
  - $T = \langle n \rangle$ for some $n \in \omega$.
    In this case, we define $A_T = \{x \in \omega^\omega \mid x(n_0) = n_1\}$, where $n \mapsto (n_0, n_1)$ is a previously fixed bijection between ω and $\omega \times \omega$.
  - $T = \bigvee_{\alpha<\tau} T_\alpha = \{\langle \bigvee, \alpha\rangle^\frown t \mid \alpha < \tau \text{ and } t \in T_\alpha\}$, where τ is an ordinal, and each T_α is in BC.
    In this case, we define $A_T = \bigcup_{\alpha<\tau} A_{T_\alpha}$.
  - $T = \neg T' = \{\langle \neg \rangle^\frown t \mid t \in T'\}$, where $T' \in \mathsf{BC}$.
    In this case, we define $A_T = \omega^\omega \setminus A_{T'}$.
- $< \Theta$-ordinal determinacy.

  Given $\lambda < \Theta$, endow λ with the discrete topology, and consider the product topology on λ^ω. Given a set of reals A and a continuous $f\colon {}^\omega\lambda \to {}^\omega\omega$, the **$(A, f)$-induced game on** λ is $\Game_\lambda(f^{-1}[A])$. Say that **$< \Theta$-ordinal determinacy** holds iff for any $\lambda < \Theta$, any continuous $f\colon {}^\omega\lambda \to {}^\omega\omega$, and any set of reals A, the (A, f)-induced game is determined.

The "official" version of the fourteenth problem is:

Problem # 14. *Does* AD *implies that all sets of reals are* ∞*-Borel.*

There seem to be two versions of the question, according to whether we assume $\mathsf{DC}_{\mathbb{R}}$ as part of the background theory. Regardless, both versions remain open, and it is also open whether $\mathsf{DC}_{\mathbb{R}}$ or ordinal determinacy follow from AD.

2015 comments.

Not much seems known. It is known that AD^+ holds in natural models of determinacy, such as models of the form $L(\mathcal{P}(\mathbb{R}))$ obtained through the derived model construction.

Woodin has shown that if $\mathsf{AD}_{\mathbb{R}}$ holds (in fact, if AD and *uniformization* hold, the latter being the statement that every binary relation on $\mathbb{R}$ can be uniformized), then all sets of reals are ∞-Borel. For this, see for instance the preprint [IW09, Theorem 4.10].

Soft arguments show that if every set of reals is ∞-Borel, and there is no uncountable sequence of distinct reals, then all sets of reals have the usual regularity property (are Ramsey, Lebesgue measurable, have the Baire property, and are either countable or contain a perfect subset). See [CK11].

This means that a positive solution to problem #14 would imply a positive solution to the long standing open question of whether AD implies that every set of reals is Ramsey.

On the other hand, in unpublished work, Woodin has shown that from the consistency of $\mathsf{ZF} + \mathsf{DC} + \mathsf{AD}+$ "Not every set of reals is ∞-Borel" one can prove the consistency of $\mathsf{ZF} + \mathsf{DC} + \mathsf{AD}+$"There exists $\kappa > \Theta$ with the strong partition property". This connects the problem with another long standing question, namely the problem of whether it is consistent to have a strong partition cardinal above Θ.

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