A review of sharps

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We hope this short note may prove useful as a guide to the general theory of sharps. Only a knowledge of the theory of $0^\sharp$ is required. This note will be updated periodically, the original version was part of the introduction to the author’s dissertation [1], written under the supervision of John Steel and Hugh Woodin at U.C. Berkeley.

**Definition 0.1.** Let $Y$ be a transitive set.

1. A class of indiscernibles for $L(Y), Y$ (informally, for $L(Y)$) is a class $I \subseteq \text{ORD}$ such that for all $\bar{a}$ elements of $Y$ and all $\alpha_1 < \cdots < \alpha_n$ and $\beta_1 < \cdots < \beta_n$ elements of $I$, for any $\varphi(\bar{x}, y_1, \ldots, y_n)$ in the language of set theory,

   \[
   L(Y) \models \varphi(\bar{a}, \bar{x}) \iff L(Y) \models \varphi(\bar{a}, \bar{\beta}).
   \]

2. Let $\varphi(t, x_1, \ldots, x_n)$ be a formula in the language of set theory, expanded with constant symbols for $Y$ and the elements of $Y$. A weak Skolem function for $\varphi$ (with respect to $L(Y), Y$) is the function $f_\varphi : {}^n L(Y) \to L(Y)$ given by

   \[
   f_\varphi(\bar{x}) = \begin{cases} 
   y & \text{if } L(Y) \models y \text{ is the unique } z \text{ such that } \varphi(z, \bar{x}); \\
   \emptyset & \text{if there is no such unique } z.
   \end{cases}
   \]

3. Let $Y \subseteq Z \subseteq L(Y)$. By $\mathcal{H}(L(Y), Z)$ we mean the closure of $Z$ under weak Skolem functions.

4. Let $I$ be a class of indiscernibles for $L(Y), Y$. We say that $I$ generates $L(Y)$ iff

   \[
   \mathcal{H}(L(Y), I \cup Y) = L(Y).
   \]

5. We say that $Y^\sharp$ exists iff there is a club proper class $I$ of indiscernibles for $L(Y), Y$ such that $I \cup Y$ generates $L(Y)$ and, moreover, for any uncountable $\eta$ such that $Y \in H_\eta$, $\mathcal{H}(L(Y), (I \cap \eta) \cup Y) = L_\eta(Y)$.

6. We say that $X^\sharp$ exists iff $Y^\sharp$ exists, where $Y = \text{Tr. Cl.(X)}$.

**Fact 0.2.** If $X \in H_\eta$ and $\eta$ is Ramsey, then $X^\sharp$ exists. \hfill $\Box$

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1We consider the language of $L(Y)$ to be expanded by constants $P_a$ for each $a \in Y \cup \{Y\}$. The natural interpretation of $P_a$ is, of course, the set $a$. 

1
The assertion “$X^\beta$ exists” refers to the existence of a proper class object. Solovay’s realization (see [2]) is that just as in the case of sharps for reals, this is in fact equivalent to the existence of a set, and it is this set we now call $X^\beta$.

**Definition 0.3.** Let $Y$ be transitive.

1. Let $\mathcal{L}_Y$ denote the language of set theory augmented with constants for the elements of $Y \cup \{Y\}$, and with $\omega$ many other constants $c_n$, $n \in \omega$, (to represent the first $\omega$ indiscernibles), and closed under terms for weak Skolem functions for formulas in the language of set theory.

2. An EM blueprint for $Y$ (EM stands for Ehrenfeucht-Mostowski) is the theory in $\mathcal{L}_Y$ of some structure $(L_\eta(Y), \in, P_n, i_n : a \in Y \cup \{Y\}, n \in \omega)$ where $Y \in H_\eta$ or $\eta = \text{ORD}$, and $(i_n : n < \omega)$ is the increasing enumeration of a set of indiscernibles for

$$(L_\eta(Y), \in, P_a)_{a \in Y \cup \{Y\}}.$$ 

3. Let $\Sigma$ be an EM blueprint for $Y$, and let $\alpha$ be an ordinal. By $\Gamma(\Sigma, \alpha)$ we mean, provided that it exists and is unique (up to isomorphism), a model $\mathcal{M}_\alpha$ such that

(a) $\mathcal{M}_\alpha \models \Sigma^*$, the restriction of $\Sigma$ to the language $\mathcal{L}_Y$ without constants for the indiscernibles.

(b) There is a set $I \subseteq \text{ORD}^{\mathcal{M}_\alpha}$ such that $(I, \in^{\mathcal{M}_\alpha}) \cong (\alpha, \in)$ which is a set of indiscernibles for $\mathcal{M}_\alpha$.

(c) $\mathcal{M}(\mathcal{M}_\alpha, I \cup \{P^{\mathcal{M}_\alpha}_a : a \in Y \cup \{Y\}\}) = \mathcal{M}_\alpha$.

4. A set of sentences $\Sigma \subseteq \mathcal{L}_Y$ is a remarkable character for $Y$ iff

(a) $\Sigma$ is an EM blueprint for $Y$. In fact, $\Sigma$ extends “$\text{ZF} + V = L(Y)$”.

(b) $\Gamma(\Sigma, \alpha)$ exists and is well-founded for all $\alpha$.

(c) For any term $t(x_0, \ldots, x_{n-1})$ in $\mathcal{L}_Y$, the sentence

$"t(c_0, \ldots, c_{n-1}) \in \text{ORD} \rightarrow t(c_0, \ldots, c_{n-1}) < c_n"$ 

belongs to $\Sigma$.

(d) For any term $t(x_0, \ldots, x_{m+n})$ in $\mathcal{L}_Y$, the sentence

$"t(c_0, \ldots, c_{m+n}) < c_n \rightarrow t(c_0, \ldots, c_{m+n}) = t(c_0, \ldots, c_{m-1}, c_{m+n+1}, \ldots, c_{m+2n+1})"$ 

belongs to $\Sigma$.

(e) $\Sigma$ satisfies the witness condition:

Whenever $\exists x \phi(x) \in \Sigma$, there is a term $t$ all of whose constants for indiscernibles already appear on $\phi(x)$, and such that $\phi(t) \in \Sigma$. 

2
The witness condition is the key condition that remarkable characters for reals (or more generally for sets of ordinals) satisfy automatically, because Skolem terms are definable in $L[x]$, $x \in \mathbb{R}$, since $L[x]$ has a definable well-ordering. Its importance lies in that it allows us to prove the following basic fact:

**Lemma 0.4 (Solovay).** If $\Sigma$ is a remarkable character for a transitive set $Y$, then

1. For all $\alpha$, the sequence $I^\alpha$ of indiscernibles of $\Gamma(\Sigma, \alpha)$ with
   \[ (I^\alpha, \in^{\Gamma(\Sigma, \alpha)}) \cong (\alpha, \in) \]
   satisfies that for any formula $\varphi(x_1, \ldots, x_n)$ in the language $\mathcal{L}_Y$,
   \[ \varphi(c_1, \ldots, c_n) \in \Sigma \]
   iff there is a $e \in \Gamma(\Sigma, \alpha)$-increasing sequence $a_1, \ldots, a_n$ of elements of $I^\alpha$ such that $\Gamma(\Sigma, \alpha) \models \varphi(a_1, \ldots, a_n)$.
2. For any cardinal $\eta$ such that $Y \in H_\eta$,
   \[ \Gamma(\Sigma, \eta) \cong L_\eta(Y). \]
3. For all $\alpha$, $I^\alpha$ is closed unbounded in $\text{ORD}^{\Gamma(\Sigma, \alpha)}$.
4. If $\alpha < \beta$, then $I^\beta$ end-extends $I^\alpha$ (seen as subsets of $\text{ORD}^{L_\eta(Y)}$ for any cardinal $\eta$ such that $\beta, Y \in H_\eta$.)
5. For any $\eta$ such that $Y \in H_\eta$,
   \[ \mathcal{H}(L(Y), I^\alpha \cup Y) = L_\eta(Y) \prec \mathcal{H}(L(Y), \bigcup_{\alpha \in \eta} I^\alpha \cup Y) = L(Y). \]
6. Let $\Sigma'$ be any remarkable character for $Y$. Then $\Sigma' = \Sigma$. □

**Corollary 0.5 (Solovay).** Let $Y$ be transitive. Then $Y^\sharp$ exists iff there is a remarkable character for $Y$. □

**Remark 0.6.** In truth, Solovay only argued these results for sharps of sets of reals (or, more precisely, for $\mathbb{R}^\sharp$), but the arguments for $0^\sharp$ lift straightforwardly.

It follows that it makes sense to define sharps in terms of the remarkable characters whose existence they guarantee:

**Definition 0.7.** Let $X$ be a set and let $Y$ be its transitive closure. Then $X^\sharp := \Sigma$, for $\Sigma$ the unique remarkable character for $Y$.

See [2], where the general notion of sharps is introduced, in the context of subsets of reals.

Notice the definition of $Y^\sharp$ is absolute in the sense that if $W \supseteq V$ is an outer model and $Y^\sharp \in V$, then $W \models (Y^\sharp)^V = Y^\sharp$.

The following is ancient, but I have been unable to find a reference:
Fact 0.8. Let $\mathbb{P}$ be a poset, and suppose $x^\sharp \in V^\mathbb{P}$, where $x$ is a real coding a set $X \in V$. Then $X^\sharp \in V$. □

It follows from the fact that Jensen’s covering lemma relativizes to all sharps, so $L[X]$ satisfies covering above $\eta$, where $X \in H_\eta$, iff $X^\sharp$ does not exist. Since set sized forcing preserves a tail of the class of cardinals, if $\mathbb{P}$ is a poset and $X^\sharp$ exists in $V^\mathbb{P}$, then $X^\sharp$ exists in $V$.

Fact 0.9 (Solovay). If $X^\sharp$ exists, then the truth sets of $L(X)$ and $L[X]$ are definable. □

The following example must be folklore, it was shown to me by Woodin. It illustrates that we cannot make do in the definition of $X^\sharp$ without the witness condition:

Recall first that after adding $\omega_1$ Cohen reals, no well-ordering of $\mathbb{R}$ belongs to $L(\mathbb{R})$. This follows immediately from the weak homogeneity of the forcing, call it $\mathbb{P}$, and the fact that $\mathbb{P}$ is ccc and $\mathbb{P} \cong \mathbb{P} \times \mathbb{P}$. From this, an elementary argument shows that, in fact, there is in $V^\mathbb{P}$ a set $\mathbb{R}_1 \subsetneq \mathbb{R}^{V^\mathbb{P}}$ and an elementary embedding $j : L(\mathbb{R}_1) \rightarrow L(\mathbb{R}^{V^\mathbb{P}})$ that fixes the ordinals, so in particular Choice fails in $L(\mathbb{R}^{V^\mathbb{P}})$ and the result follows.

Claim 0.10. Let $V = L[\mu]$ be the smallest inner model for a measurable cardinal and let $G$ be $\operatorname{Add}(\omega, \omega_1)$-generic over $V$. Then

1. $(\mathbb{R}^\sharp)^{V[G]}$ exists.
2. $(\mathbb{R}^\sharp)^{V[G]} \cap V \in V$.
3. $(\mathbb{R}^\sharp)^{V[G]} \cap V$ satisfies conditions 4.(a)–(d) of Definition 0.3 for $(\mathbb{R}^\sharp)^V$.

□

If we could dispense with the witness condition in Definition 0.3, it would follow from the claim that $\mathbb{R}^{L[\mu]}$ is not well-orderable by a well-ordering in $L(\mathbb{R}^{L[\mu]})$. This is absurd, since in fact $\mathbb{R}^{L[\mu]}$ admits a $\Delta_3$-well-ordering.

Remark 0.11. Of course, the same arguments generalize to larger sharp-like objects, like daggers or pistols.

The theory of sharps is usually recalled in connection with finestructural arguments. In this context, $X^\sharp$ is usually defined as a particular kind of mouse.

Fact 0.12. Let $X$ be a set. Then $X^\sharp$ exists iff there is an active $X$-mouse. □

There is therefore no lack of generality in using this approach. We actually obtain quite more information than what was stated in Fact 0.12. For example, by standard techniques a mouse as in 0.12 is unique if it exists, and so we can identify it with $X^\sharp$. Moreover, for example if $x \in \mathbb{R}$, $x^\sharp$ and the minimal active $x$-mouse share the same Turing degree.
References
