Real-valued Measurable Cardinals and Σ^2_1-Definable Well-orderings of the Reals

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Abstract

We announce and present a sketch of a result of Woodin that obtains (from a proper class of Woodin cardinals) the Ω-consistency of the statement

The continuum is real-valued measurable, and there is a Σ^2_1-well-ordering of $\mathbb{R}$.

This is a specific instance of an Ω-consistent statement for which it is open whether it is forceable. The result depends on a comparison argument for AD^+-models. Real-valued measurability is treated in terms of elementary embeddings.

I. Real-valued Measurability

Definition 1 (Ulam). A cardinal κ is *real-valued measurable*, $\mathsf{RVM}(\kappa)$, iff there is a κ-additive probability measure ν with domain $\mathcal{P}(\kappa)$ which is null on singletons. We call ν a *witnessing* probability.

A real-valued measurable cardinal κ is *atomlessly measurable* iff there is an atomless witnessing probability ν.

Fact 2. *If* $\mathsf{RVM}(\kappa)$*, then* κ *is either measurable or atomlessly measurable, in which case* $\kappa \leq \mathfrak{c}$*.*

$\square$

Theorem 3 (Solovay). $\mathsf{RVM}(\kappa)$ *iff there is* $\lambda \geq \omega_1$ *such that*

$$V^{\mathtt{Random}_\lambda} \models \exists j : V \xrightarrow{\prec} N, \quad \mathrm{cp}(j) = \kappa,$$

where $\mathtt{Random}_\lambda$ *is the forcing for adding* λ *many random reals.*

If $\kappa \leq \mathfrak{c}$ *and* $\mathsf{RVM}(\kappa)$*, then in addition we can require that* $V^{\mathtt{Random}_\lambda} \models {}^\omega N \subseteq N$*.* $\square$

II. Random Forcing

Definition 4. Let $\mathbb{B}$ be a σ-complete Boolean algebra. A *'probability measure'* on $\mathbb{B}$ is a function $\nu : \mathbb{B} \to [0, 1]$ such that

1. $\nu(a) = 0$ iff $a = \mathtt{0}$.
2. $\nu(\mathtt{1}) = 1$.
3. ν is σ-additive: If $\{ a_n : n \in \omega \}$ is an antichain in $\mathbb{B}$, so $a_n \cdot a_m = \mathtt{0}$ whenever $n \neq m$, then

$$\nu\Big(\sum_n^{\mathbb{B}} a_n\Big) = \sum_n \nu(a_n).$$

We call $(\mathbb{B}, \nu)$ a *measure algebra.*

Definition 5. Give 2^λ the product topology of λ copies of the discrete space 2.

1. $\mathcal{B}_\lambda$ is the class of Borel subsets of 2^λ.
2. $\varphi = \varphi_\lambda$ is the product measure on 2^λ.
3. $\mathcal{N}_\varphi$ is the ideal of φ-null subsets of $\mathcal{B}_\lambda$.
4. $\texttt{Random}_\lambda = \mathcal{B}_\lambda / \mathcal{N}_\varphi$.

$\texttt{Random}_\lambda$ is a σ-complete Boolean algebra.

Fact 6. *1. For all λ, $\texttt{Random}_\lambda$ is ccc. Thus $\texttt{Random}_\lambda$ is a complete Boolean algebra.*

2. The map $\nu : \texttt{Random}_\lambda \to [0,1]$ given by $\nu([X]) = \varphi(X)$, where φ is as described above and $[X]$ denotes the equivalence class of the Borel subset $X \subseteq 2^\lambda$, is a 'probability measure', so $(\texttt{Random}_\lambda, \nu)$ is a measure algebra. $\square$

Definition 7. 1. For $\mathbb{B}$ a complete Boolean algebra, the *generating number* of $\mathbb{B}$ is $\tau(\mathbb{B}) := \min\{\, |X| : X \text{ generates } \mathbb{B} \text{ (as a complete algebra)} \,\}$.

2. $\mathbb{B}$ is *τ-homogeneous* iff[a] $\tau(\mathbb{B}) = \tau(\mathbb{B}{\restriction}_a)$ for any $a \neq \mathtt{0}$.

Fact 8. *1. If $\mathbb{B}$ is a complete measure algebra which is homogeneous in the forcing sense, then $\mathbb{B}$ is τ-homogeneous.*

2. $\mathtt{Random}_\lambda$ is homogeneous. Thus, it is τ-homogeneous, and $\tau(\mathtt{Random}_\lambda) = \lambda$. □

Theorem 9 (Maharam). *If $\mathbb{B}$ is a complete τ-homogeneous measure algebra, then it is isomorphic as a measure algebra to exactly one $\mathtt{Random}_\lambda$ up to the cardinality of λ.* □

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[a] For $p \in \mathbb{B} \setminus \{\mathtt{0}\}$, $\mathbb{B}{\restriction}_p$ is the Boolean algebra of elements of $\mathbb{B}$ below p.

Fact 10. *If* $\mathbb{B} \lessdot \mathtt{Random}_\lambda$ *(i.e.,* $\mathbb{B}$ *is a complete subalgebra of* $\mathtt{Random}_\lambda$*), then* $\mathbb{B} \cong \mathtt{Random}_\gamma$ *for some* γ*.* $\square$

Notice that, conversely, if $\gamma < \lambda$, then $\mathtt{Random}_\gamma \lessdot \mathtt{Random}_\lambda$.

Fact 11. *Let* $\mathbb{B} \lessdot \mathtt{Random}_\lambda$*. Then* $\mathtt{1} \Vdash_{\mathbb{B}} (\mathtt{Random}_\lambda)^V / \dot{\mathbb{B}} \cong \mathtt{Random}_\gamma$ $\mathtt{for\ some}$ γ*.* $\square$

Fact 12. *If* $W \supseteq V$ *is an outer model and* G *(identified as a subset of* λ*) is* $(\mathtt{Random}_\lambda)^W$*-generic over* W*, then* G *is* $(\mathtt{Random}_\lambda)^V$*-generic over* V*.* $\square$

This follows from the results of [Ku1], §3.

If (X, ν, Σ) is a measure space, $\mathcal{N}_\nu$ is the ideal generated by the ν-null sets.

Fact 13 (C.). *Suppose* $\mathsf{RVM}(\kappa)$. *There is a witness* ν *such that* $\mathbb{B}_\nu = \mathcal{P}(\kappa)/\mathcal{N}_\nu$ *is homogeneous (so* $\mathbb{B}_\nu = \texttt{Random}_\lambda$ *for some* λ.*) Let* G *be* $\mathbb{B}_\nu$*-generic over* V*, and in* $V[G]$ *let* $j : V \to N$ *be the associated generic embedding. Then the forcing* $j(\texttt{Random}_\kappa)/\texttt{Random}_\kappa$ *is isomorphic (in* $V[G]$*, not simply in* N*) to* $\texttt{Random}_{j(\kappa)}$. $\square$

Notice the particular case where κ is measurable, so $\mathbb{B}_\nu$ is trivial and $G \in V$.

III. Σ^2_n-definability

Fact 14. *Let $\varphi(\vec{x})$ be a Σ^2_1-formula. Then there is ψ, and a transitive structure $M \models \mathsf{ZFC}^{-\varepsilon}$ such that $\mathbb{R} \subseteq M$, $|M| = \mathfrak{c}$, or even ${}^{\omega}M \subseteq M$, such that for all reals $\vec{r}$,*

$$\varphi(\vec{r}) \iff M \models \psi(\vec{r}).$$

Here, $\mathsf{ZFC}^{-\varepsilon}$ is a sufficiently strong fragment of ZFC. [For example, we can take $\mathsf{ZFC}^{-\varepsilon}$ to mean $\mathsf{ZFC}^{-} + \mathcal{P}(\mathbb{R})$ exists, or $\mathsf{ZFC}\restriction_{\Sigma_{200}}$, i.e., ZFC with replacement restricted to Σ_{200}-statements.] $\square$

Remark 15. In fact, the pointclass Σ^2_n can be identified by this method with the class $\Sigma_n(H_{\mathfrak{c}^+}, \in, H_{\omega_1})$, where H_{ω_1} is seen as a parameter and therefore quantification over it is considered bounded.

IV. $L(\mathbb{R})$, $\mathsf{RVM}(\mathfrak{c})$ and Definable Well-orderings

We are interested in definable well-orderings of $\mathbb{R}$ in the presence of real-valued measurability. The focus on the pointclass Σ^2_1 is due to the following fact.

Theorem 16 (C.). *If $\kappa \leq \mathfrak{c}$ and $\mathsf{RVM}(\kappa)$ then no well-ordering of $\mathbb{R}$ belongs to $L(\mathbb{R})$.*

Proof: Assume that $\kappa \leq \mathfrak{c}$, $\mathsf{RVM}(\kappa)$ and there is $\varphi(x, y, z)$ such that for some real t the relation $\{ (r, s) : L(\mathbb{R}) \models \varphi(r, s, t) \}$ is a well-ordering of $\mathbb{R}$.

Let λ be such that in $V^{\mathtt{Random}_\lambda}$ there is an embedding $j : V \xrightarrow{\prec} N$ such that $\mathrm{cp}(j) = \kappa$ and ${}^\omega N \subseteq N$. Then $j{\restriction}_{L(\mathbb{R})^V} : L(\mathbb{R})^V \xrightarrow{\prec} L(\mathbb{R})^{V^{\mathtt{Random}_\lambda}}$, so $\varphi(x, y, t)$ still defines a well-ordering of $\mathbb{R}^{V^{\mathtt{Random}_\lambda}}$. Contradiction. $\square$

V. $L[\mu]$ and Σ^2_1-well-orderings

From now on, all results mentioned are due to Woodin.

Recall:

Definition 17. By $L[\mu]$ we mean the smallest proper class inner model of the theory

$$\text{ZFC} + \text{There exists a measurable.}$$

In this context, by μ we always mean a witness to measurability, i.e.,

$$L[\mu] \models \quad \mu \text{ is a normal } \kappa\text{-complete measure on some cardinal } \kappa.$$

We abuse notation and talk about the theory $V = L[\mu]$.

Assume κ is measurable, GCH holds below κ, and let $j : V \to N$ be a normal ultrapower embedding with $\mathrm{cp}(j) = \kappa$.

Let $\mathbb{Q} = \mathtt{Random}_\kappa$ and $\mathbb{P}$ be the Easton product over the inaccessibles $\lambda < \kappa$ of $\mathtt{Add}(\lambda^+, 1) \times \mathtt{Add}(\lambda^{++}, 1)$.

Let $G_{\mathbb{P}} \times G_{\mathbb{Q}}$ be $\mathbb{P} \times \mathbb{Q}$ generic over V. Then $j(\mathbb{Q})/\mathbb{Q}$ is isomorphic to an appropriate Random forcing in any intermediate model between $V[G_{\mathbb{Q}}]$ and $V_1 := V[G_{\mathbb{Q}}][G_{\mathbb{P}}]$.

Notice that $\mathbb{R}^{V[G_{\mathbb{Q}}]} = \mathbb{R}^{V_1}$.

Claim 18. *There is* $G^* \in V$ *such that:*

- *If* H *is* $j(\mathbb{Q})/\mathbb{Q}$*-generic over* V_1 *then, in* $V_1[H]$*,* j *lifts to* $j_2 : V_1 \to N[G_{\mathbb{P}}][G^*][G_{\mathbb{Q}}][H]$*, and therefore (by Solovay's theorem)* $\mathsf{RVM}(\mathfrak{c})$ *holds in* V_1*.*

- *The restriction of* j_2 *to* $V[G_{\mathbb{P}}]$ *is* $j_1 : V[G_{\mathbb{P}}] \to N[G_{\mathbb{P}}][G^*]$ *(so* κ *is still measurable), and if* $j(\mathbb{P}) = \mathbb{P} \times \mathbb{P}_{tail}$*, then* G^* *is* $\mathbb{P}_{tail}$*-generic over* $N[G_{\mathbb{P}}]$*.*

- *Similarly, the restriction of* j_2 *to* $V[G_{\mathbb{Q}}][H]$ *is* $j_3 : V[G_{\mathbb{Q}}] \to N[G_{\mathbb{Q}}][H]$ *and witnesses* $\mathsf{RVM}(\mathfrak{c})$ *in* $V[G_{\mathbb{Q}}]$*.* $\square$

In $V[G_{\mathbb{Q}}]$, let $A \subset \kappa$ code a well-ordering of $\mathbb{R}$ in order type κ.

Let $\langle\, \delta_\alpha : \alpha < \kappa \,\rangle$ be the increasing enumeration of the inaccessibles in V below κ. For $\alpha < \kappa$, let G_α be the α^{th} component of $G_{\mathbb{P}}$, so G_α is the product of an $\mathtt{Add}(\delta_\alpha^+, 1)$-generic and an $\mathtt{Add}(\delta_\alpha^{++}, 1)$-generic over V. Let G_α^* be the $\mathtt{Add}(\delta_\alpha^+, 1)$-generic, if $\alpha \in A$, and the $\mathtt{Add}(\delta_\alpha^{++}, 1)$-generic, if $\alpha \notin A$. Finally, let

$$g = \prod_{\alpha<\kappa} G_\alpha^*.$$

Notice A is definable from g.

An easy argument shows that $j_2(g)$ is definable from G^* and $j_3(A)$ (Recall $G^* \in V$ and $j_3(A) \in V[G_{\mathbb{Q}}][H]$). It follows that $\mathfrak{c} = \kappa$ is real-valued measurable in $V[G_{\mathbb{Q}}][g]$, and that a lifting of j to $j^* : V[G_{\mathbb{Q}}][g] \to N[G_{\mathbb{Q}}][H][j_2(g)]$ definable in $V[G_{\mathbb{Q}}][g][H]$ serves as a witness.

Theorem 19 (Woodin). *If* $V = L[\mu]$ *then, in* $V[G_{\mathbb{Q}}][g]$, $\mathsf{RVM}(\mathfrak{c})$ *and there is a* Δ^2_1*-well-ordering of* $\mathbb{R}$.

Proof: Let $V_1 = L[\mu][G_{\mathbb{Q}}][g]$, so $\mathbb{R}^{V_1} = \mathbb{R}^{L[\mu][G_{\mathbb{Q}}]}$ and $\mathsf{RVM}(\mathfrak{c})$ holds in V_1. We claim that the well-ordering coded by A is Σ^2_1 in V_1. This we verify by "guessing" the ground model.

Definition 20. Let N be a transitive structure which models enough set theory. We say that N *satisfies countable covering* if and only if

$$\forall \sigma \in \mathcal{P}_{\omega_1}(N)\, \exists \tau \in N\, (\sigma \subseteq \tau \text{ and } N \models |\tau| \leq \aleph_0).$$

Claim 21. *In V_1, suppose M is transitive, $|M| = \mathfrak{c}$, $M \models \mathsf{ZFC}^{-\varepsilon} + V = L[\mu]$. Let κ_M be the measurable cardinal in the sense of M, and $\kappa = \mathfrak{c}$. Suppose $\kappa_M \geq \mathfrak{c}$, M is iterable and satisfies countable covering.*

Then $M_\kappa = L[\mu]_\kappa$.[a]

The point is that we have identified the part of the ground model relevant to our argument in a projective fashion.

Let K^{DJ} be the Dodd-Jensen core model.

[a]The notation we use here is ambiguous. For N a model and α an ordinal, $N_\alpha = \{ x \in N : \mathrm{rk}\,(x) < \alpha \}$, where $\mathrm{rk}\,(x)$ is the set-theoretic rank of x. In particular, $L[\mu]_\kappa$ is *not* the κ^{th}-stage in the (classical) constructible hierarchy of $L[\mu]$.

Proof: An initial segment of $L[\mu]$ satisfies the requirements: Iterability is clear, and countable covering holds because $\mathbb{Q}$ is ccc and $\mathbb{P}$ is ω_1-closed.

Assume M satisfies the requirements of the Claim. Suppose $M \not\supseteq L[\mu]_\kappa$, and let $a \in L[\mu]_\kappa \setminus M$.

Notice that (provably in $\mathsf{ZFC} + L[\mu]$ exists), $K^{DJ}_\kappa = L[\mu]_\kappa$. Hence, there is a mouse $\bar{M} = L[\bar{\mu}]_{\bar{\alpha}}$ with $a \in \bar{M}_{\bar{\kappa}}$, where $\bar{\mu}$ is a $\bar{M}$-measure on $\bar{\kappa}$. Since $a \notin M$ and M does not move in the comparison of $\bar{M}$ with M, $\bar{M} \supseteq M$.

We must in fact have equality or the critical sequence (of $\bar{M}$) would violate covering since $\bar{\kappa} < \kappa$.

The other containment is clear. $\triangle$

We are basically done now: To require iterability of a model M as in the Claim is a projective requirement (for example, if $M \models V = K^{DJ}$, iterability of M states that every countable mouse required to verify $V = K^{DJ}$ is iterable, and this iterability is in turn a Π^1_2-statement.) Hence, to define A in a Σ^2_1-way it suffices to notice the following Claim.

Claim 22. *In V_1 suppose $\hat{\delta} < \kappa$ and $a \subseteq \hat{\delta}^+$ is such that $a \notin L[\mu]$ is $\texttt{Add}(\hat{\delta}^+, 1)$-generic over $L[\mu]$. Then $\hat{\delta}$ is an inaccessible or the successor of an inaccessible cardinal δ_β, and $\beta \in A$ iff $\hat{\delta} = \delta_\beta$.*
$\triangle$

It follows that A can be defined by refering to those cardinals $\hat{\delta}$ for which there is a set a as above.

Now observe that in $V[G_{\mathbb{Q}}][g]$ we can define A as follows:

Let $\Psi(M)$ denote the conjunction of:

> $M \models \mathsf{ZFC}^{-\varepsilon} + V = L[\mu]$, $|M| = \mathfrak{c}$, M is transitive, iterable and satisfies countable covering, and $\kappa^M \geq \mathfrak{c}$.

Notice that $\Psi(M)$ is a projective assertion about M.

Moreover, for any M such that $\Psi(M)$ holds and any $\alpha < \mathfrak{c}$, $\alpha \in A$ iff, letting $\hat{\delta}_\alpha$ denote the α^{th} inaccessible in M, there is $a \notin M$ which is $\mathtt{Add}(\hat{\delta}^+_\alpha, 1)$-generic over $L[\mu]$.

For x, y reals, let $\psi(x, y)$ hold iff

> There is M such that $\Psi(M)$ holds and in A (defined as above) x is coded before y.

Then ψ is a Σ^2_1-well-ordering of $\mathbb{R}^{V[G_{\mathbb{Q}}][g]}$. $\square$

VI. A Generalization

Notice essentially the same argument provides models with a Σ^2_1-well-ordering of $\mathbb{R}$ where $\mathsf{RVM}(\mathfrak{c})$ holds, as long as the ground model is fine structural, and the iterability condition is projective.

Granting large cardinals, and starting with a definable fine structural model, the construction produces a model of $\mathsf{RVM}(\mathfrak{c})$ together with a $\Sigma^2_1(\mathsf{UB})$-well-ordering of $\mathbb{R}$. Here, $\Sigma^2_1(\mathsf{UB})$ is the pointclass of sets of reals A such that for some projective formula ψ and some real parameter r, A can be defined by: For all $s \in \mathbb{R}$,

$$s \in A \quad \Longleftrightarrow \quad \exists B\, (\psi(s, r, B) \text{ and } B \in \mathsf{UB}),$$

and UB is the pointclass of all Universally Baire sets of reals.

VII. Ω-logic

Recall that if M is a transitive structure, $M_\alpha = \{\, x \in M : \mathrm{rk}\,(x) < \alpha \,\}$.

Definition 23 (Ω-logic). Let $T \supseteq \mathsf{ZFC}$ and let ϕ be a sentence. Then

$$T \models_\Omega \phi$$

iff for all $\mathbb{P}$ and all λ, if $V_\lambda^{\mathbb{P}} \models T$, then $V_\lambda^{\mathbb{P}} \models \phi$.

A logic satisfying this property is said to be *generically sound.*

The following is immediate:

Fact 24. *The statement* $\mathsf{RVM}(\mathfrak{c})+$ There is a Δ^2_1-well-ordering of $\mathbb{R}$ *can be rendered in a* Δ_2*-way.* $\square$

Remark 25. An Ω*-satisfiable* sentence ϕ, i.e., a sentence ϕ such that $\neg\phi$ is not Ω-valid, is one such that for some $\mathbb{P}$ and α, $V^{\mathbb{P}}_\alpha \models \mathsf{ZFC} + \phi$. Taking for granted that there are enough ordinals α such that $V_\alpha \models \mathsf{ZFC}$, so this discussion is not vacuous, if ϕ is Σ_2 and Ω-satisfiable, then in fact ϕ is forceable *over* V, i.e., for some $\mathbb{P}$, $V^{\mathbb{P}} \models \phi$.

Definition 26. By *our Base Theory* we mean

ZFC + There is a proper class of Woodin cardinals.

Theorem 27 (Generic Invariance). *Assume our Base Theory. Let $T \supseteq \mathsf{ZFC}$ and let ϕ be a sentence. Then $T \models_\Omega \phi$ iff for all $\mathbb{P}$,*
$V^{\mathbb{P}} \models T \models_\Omega \phi$. □

Recall:

Definition 28 (Feng, Magidor, Woodin [FeMaW]). A set $A \subseteq \omega^\omega$ is λ-*Universally Baire* iff there are λ-*absolutely complementing* trees for A, i.e., trees T, T^* on $\omega \times X$ for some X, such that

1. $A = p[T]$ and $\omega^\omega \setminus A = p[T^*]$.
2. $\mathtt{1} \Vdash_{\mathbb{P}} p[T] \cup p[T^*] = \omega^\omega$ for any forcing $\mathbb{P}$ of size at most λ.

A is ∞-Universally Baire or, simply, Universally Baire, iff it is λ-Universally Baire for all λ.

Notice that if A is λ-Universally Baire, and $T, T^*, \mathbb{P}$ are as above, then $\mathtt{1} \Vdash_{\mathbb{P}} p[T] \cap p[T^*] = \emptyset$.

The Universally Baire sets generalize the Borel sets and have all the usual regularity properties. They form a σ-algebra and are closed under continuous reducibility. They are closed under more generous versions of reducibility under reasonable large cardinal assumptions. For example:

Fact 29. *Assume our Base Theory. Suppose A is Universally Baire. Then $A^\sharp$ is Universally Baire. Equivalently, every set of reals in $L(A, \mathbb{R})$ is Universally Baire.* $\square$

A fundamental connection between Universal Baireness and determinacy is given by the following result:

Theorem 30 (Neeman). *Suppose A is Universally Baire and there is a Woodin cardinal. Then A is determined.* □

Here is a sample application that we will have occasion to use:

Corollary 31. *Assume our Base Theory. Suppose A is Universally Baire. Then* $L(A, \mathbb{R}) \models \mathsf{AD}^+$. □

Definition 32. Let A be Universally Baire. Let $\mathbb{P}$ be a forcing notion, and let G be $\mathbb{P}$-generic over V. Then the interpretation A_G of A in $V[G]$ is

$$A_G = \bigcup \{ p[T] : T \in V \text{ and } V \models A = p[T] \}.$$

This is the natural notion we would expect: If T, T^* are λ-complementing trees such that $p[T] = A$, if $|\mathbb{P}| \leq \lambda$ and G is $\mathbb{P}$-generic over V, then $V[G] \models A_G = p[T]$.

Definition 33. Let $A \subseteq \omega^\omega$ be Universally Baire. A transitive set $M \models \mathsf{ZFC}$ is *A-closed* iff for all $\mathbb{P} \in M$ and all G, $\mathbb{P}$-generic over V,

$$V[G] \models A_G \cap M[G] \in M[G].$$

Fact 34. *Assume our Base Theory. Let A be a Universally Baire set. Then there are A-closed countable transitive models of* ZFC. $\square$

Definition 35 ($\vdash_\Omega$). Let $T \supseteq \mathsf{ZFC}$ be a theory, and let ϕ be a sentence. Then

$$T \vdash_\Omega \phi$$

iff there exists a Universally Baire set A such that

1. $L(A, \mathbb{R}) \models \mathsf{AD}^+$.
2. $A^\sharp$ exists and is Universally Baire.
3. Whenever M is a countable, transitive, A-closed model of ZFC and $\alpha \in \mathrm{ORD}^M$ is such that $M_\alpha \models T$, then $M_\alpha \models \phi$.

Theorem 36 (Generic Invariance). *Assume our Base Theory. Let $T \supseteq \mathsf{ZFC}$ and let ϕ be a sentence. Then $T \vdash_\Omega \phi$ iff for all $\mathbb{P}$, $V^{\mathbb{P}} \models T \vdash_\Omega \phi$.*
$\square$

Theorem 37 (Generic Soundness). *Let $T \supseteq \mathsf{ZFC}$ and let ϕ be a sentence. Suppose $T \vdash_\Omega \phi$. Then $T \models_\Omega \phi$.* $\square$

The Ω-conjecture is the statement that $\vdash_\Omega$ is the notion of provability associated to $\models_\Omega$ in the sense that the completeness theorem for Ω-logic holds.

Conjecture 38 (Ω-Conjecture). *Assume our Base Theory and let ϕ be a sentence. Then* $\mathsf{ZFC} \models_\Omega \phi$ *iff* $\mathsf{ZFC} \vdash_\Omega \phi$.

Woodin has shown that the Ω-conjecture is true unless (in a precise sense) there are large cardinal hypothesis implying a strong failure of iterability, see [W] and [W1].

Let φ be Ω-*consistent* iff $\neg\varphi$ is not Ω-provable. Under our Base Theory this is equivalent to asserting that for all A Universally Baire there is an A-closed M and an $\alpha \in \mathrm{ORD}^M$ such that $M_\alpha \models \mathsf{ZFC} + \varphi$.

Theorem 39 (Woodin). *Let φ be the statement* $\mathfrak{c}$ is real-valued measurable and there is a Σ^2_1-well-ordering of $\mathbb{R}$. *Assume our Base Theory. Then φ is Ω-consistent.*

Corollary 40. *Assume our Base Theory and the Ω-conjecture. Let φ be as above. Then φ is forceable over V.* $\square$

VIII. AD^+

The proof of Woodin's result requires that we be able to identify (in a Σ^2_1-fashion) fine structural ground models for which no straightforward thinness assumption, like $V = L[\mu]$, is available, and leads us to consider comparison arguments for models of AD^+.

AD^+ is a technical strengthening of AD, due to Hugh Woodin.

It intends to axiomatize those sentences φ such that whenever $M \subseteq N$ are transitive models of $\mathsf{ZF}^{-\varepsilon} + \mathsf{AD}$ with the same reals and such that every set of reals in M is Suslin[a] in N, then $M \models \varphi$.

[a] A set $A \subseteq \mathbb{R}$ is *Suslin* iff it is κ-Suslin for some cardinal κ, which is to say that there is a tree T on $\omega \times \kappa$ whose projection is A.

The assumption on “external Susliness” of the sets of reals in M guarantees that they possess (descriptive set-theoretic) scales (in N). The intuition is that many consequences of AD depend on the existence of scales for different sets of reals, but an examination of the arguments tends to show that Susliness itself need not hold in the model. This is why AD^+ was originally known as “AD within scales”, and some authors still refer to it in this way.

The usual motivation for AD^+ is less technical: AD^+ intends to lift to models $M = L(\mathcal{P}(\mathbb{R}))^M$ the rich theory that $L(\mathbb{R})$ satisfies under the assumption of $\mathsf{AD}^{L(\mathbb{R})}$.

Definition 41 (Woodin). Assume ZF. AD^+ is the following theory:

1. $\mathsf{DC}_{\mathbb{R}}$.
2. Every set of reals is ∞-*Borel*. This is to say: For all $A \subseteq \mathbb{R}$ there is an ordinal α, a set of ordinals S, and a formula $\phi(x, y)$ such that
$$A = \{\, r \in \mathbb{R} : L_\alpha[S, r] \models \phi(S, r) \,\}.$$
3. Suppose $\lambda < \Theta$, where $\Theta := \sup\{\, \alpha : \exists f : \mathbb{R} \to \alpha \,(f \text{ is onto}) \,\}$. Endow λ with the discrete topology, and $\lambda^\omega := {}^\omega\lambda$ with the product topology. Assume $\pi : \lambda^\omega \to \omega^\omega$ is continuous. Then for each $A \subseteq \mathbb{R}$, the set $\pi^{-1}``A$ is determined.

Thus, AD^+ strengthens AD.

The most important open question in the theory of determinacy is usually understood as the conjunction of three questions, one for each requirement in Definition 41, *even* if the base theory in the last two questions is assumed to be $\mathsf{ZF} + \mathsf{DC}_{\mathbb{R}}$.

Question 42 (Woodin). *Assume* ZF. *Are* AD *and* AD^+ *equivalent?*

The answer is *yes* for all known models of AD.

That the given axiomatization of AD^+ has indeed succeeded in capturing the intuitive notion it intended to is the content of the following result:

Theorem 43. *Let $M \subseteq N$ be transitive models of $\mathsf{ZF}^{-\varepsilon} + \mathsf{AD}$ such that $\mathbb{R}^M = \mathbb{R}^N$ and every set of reals in M is Suslin in N. Then $M \models \mathsf{AD}^+$.* □

The following is not really an AD^+-result, but AD^+ builds on analogues of it for models larger than $L(\mathbb{R})$:

Theorem 44. $L(\mathbb{R}) \models \exists S \subseteq \Theta\,(\mathsf{HOD} = L[S])$.
□

The set S as in 44 is obtained by a version of Vopenka's forcing due to Woodin that can add $\mathbb{R}$ to $\mathsf{HOD}^{L(\mathbb{R})}$. Variants of this forcing are very useful at different points during the development of the AD^+ theory, and we will have ocassion to use below the following AD^+-version of this result:

Theorem 45. *Suppose* $\mathsf{AD}^+ + V = L(\mathcal{P}(\mathbb{R}))$ *holds. Then there is* $S \subseteq \Theta$ *such that* $\mathsf{HOD} = L[S]$. $\square$

S can be taken to code the Σ_1-theory of Θ in $L(\mathcal{P}(\mathbb{R}))$. If $V = L(A, \mathbb{R})$ for some set $A \subseteq \mathrm{ORD}$, then S can be obtained by a generalization of the version of Vopenka's forcing hinted at above.

Solovay's Basis Theorem, a very important consequence of AD in $L(\mathbb{R})$, also generalizes to the AD^+ context:

Theorem 46. *Assume* $\mathsf{AD}^+ + V = L(\mathcal{P}(\mathbb{R}))$. *Then*

1. Σ^2_1 *has the scale property,*
2. *Whenever* $A \subseteq \mathbb{R}$ *is* Σ^2_1, *there is a tree* $T \in \mathsf{HOD}$ *such that* $A = p[T]$, *and*
3. *Every true* Σ_1 *sentence is witnessed by a (set coded by a)* $\underset{\sim}{\Delta}^2_1$ *set of reals.* $\square$

The following theorem is the key argument in the proof of Woodin's theorem 39. We aim to build a set sized model which will code a Σ^2_1-well-ordering of $\mathbb{R}$. That the well-ordering has the claimed complexity follows from our capacity to identify enough of this model in a Σ^2_1-way. This is precisely what the following theorem entails. Let $(\star\star)$ denote the following statement:

> Suppose M and N are inner models of AD^+, $\mathbb{R} \subset M \cap N$, and suppose the sets of reals in $M \cup N$ are ω_1-Universally Baire. Then the Suslin-Cosuslin sets of M and N are Wadge-comparable. In particular, their Σ^2_1-facts are comparable.

Theorem 47 (Woodin). *Suppose that $\mathcal{I}$ is a normal ideal on ω_1 such that*

$$\mathcal{P}(\omega_1)/\mathcal{I} \equiv \texttt{Coll}(\omega, \omega_1) \times \texttt{Random}_{\mathfrak{c}}.$$

Then $(\star\star)$ holds. $\square$

A few remarks are in order. First, the argument is fine structural in nature, and uses a comparison lemma involving hybrid structures. In this sense, it is strongly reminiscent of the following, earlier, result, which is part of the proof that the existence of ω_1-dense ideals on ω_1 implies $\mathsf{AD}^{L(\mathbb{R})}$:

Theorem 48 (Woodin). *Assume there is an ω_1-dense ideal. Then* $(\star\star)$ *holds.* $\square$

Second, and more importantly, by using a new fine structural hierarchy and a companion mouse set theorem the details of which remain unpublished, Woodin has obtained, via a much more elaborate argument, a proof of the following result:

Theorem 49 (Woodin). $(\star\star)$ *holds.* $\square$

The proof of 39 requires the construction of A-closed models of AD^+ for different Universally Baire sets A. We also need these models to be identifiable in a Σ_1^2-way. This we will accomplish, thanks to the Basis Theorem, via the following lemma:

Lemma 50. *Assume* AD^+. *If* $A \subseteq \mathbb{R}$ *is* Δ_1^2 *then there is* A^* *such that* $A \times A^*$ *is* Δ_1^2 *in* $L(A \times A^*, \mathbb{R})$. $\square$

We will use the lemma to produce a sequence of sets. To ensure that they are Δ^2_1 in the appropriate model, we will make use of the following two facts:

Fact 51. *Assume* AD^+*. Let* M *be an inner model of* AD^+ *that contains all the reals. Let* $A \subseteq \mathbb{R}$ *be such that* $A \in M$ *and* $M \models A \text{ is } \Delta^2_1$*. Then* A *is* Δ^2_1*.* $\square$

It follows that if A is Δ^2_1 in some inner model of AD^+, then it is Δ^2_1 in all larger inner models of AD^+ with the same reals.

Fact 52. *Assume* AD^+ *and let* A *be* Δ^2_1*. Suppose that* $A^\sharp$ *exists. Then* $A^\sharp$ *is* Δ^2_1*.* $\square$

IX. A remark on $\mathbb{Q}_{\max}$

It would take us too long to define $\mathbb{Q}_{\max}$ (see [W]). It suffices to say that $\mathbb{Q}_{\max}$ is a forcing notion *definable* in $L(\mathbb{R})$ (under $\mathsf{AD}^{L(\mathbb{R})}$) such that when applied to suitable models of AD^+ produces extensions satisfying ZFC where NS_{ω_1} is an ω_1-dense ideal.

The following is contained in [W], Theorems 6.73 and 6.74:

Theorem 53. *Assume* $\mathsf{AD}^{L(\mathbb{R})}$. *Then* $\mathbb{Q}_{max}$ *is* ω*-closed. Let* G *be* $\mathbb{Q}_{max}$*-generic over* V. *Then*

1. $L(\mathbb{R})[G] \models \mathsf{AC}$. *In fact,* $H_{\omega_2}^{L(\mathbb{R})[G]}$ *satisfies Woodin's* ϕ_{AC}.

2. $\mathcal{P}(\omega_1)^{V[G]} \subseteq L(\mathbb{R})[G]$.

3. NS_{ω_1} *is* ω_1*-dense.* $\Box$

Most of Theorem 53 generalizes more or less straightforwardly:

Let $N \models \mathsf{AD}^+ + V = L(\mathcal{P}(\mathbb{R})) + \Theta = \theta_0$ be an inner model, $\mathbb{R} \subseteq N$. Then $\mathsf{AD}^{L(\mathbb{R})}$, so $\mathbb{Q}_{\max}$ is ω-closed. Let G be $\mathbb{Q}_{\max}$-generic over N. Then 2. and 3. of Theorem 53 still hold. However, Choice itself might be problematic, but we have

$$N[G] \models \omega_1\text{-}\mathsf{DC}.$$

X. The Ω-consistency of Σ^2_1-well-orderings, and Real-valued Measurables

Recall the statement of Theorem 39:

Theorem 39 (Woodin). *Assume Our Base Theory. Then it is Ω-consistent that $\mathfrak{c}$ is real-valued measurable and there is a Σ^2_1-well-ordering of $\mathbb{R}$.*

If we relax the conclusion, then there is no need to appeal to Ω-logic:

Theorem 54 (C.). *Assume κ is measurable and* GCH *holds. There is a forcing extension where $\kappa = \mathfrak{c}$ is real-valued measurable, and there is a Σ^2_2-well-ordering of $\mathbb{R}$.* $\square$

However, the following remains open even if the Ω-conjecture holds:

Question 55. *Assume κ is measurable and* GCH *holds. Is there a forcing extension where $\kappa = \mathfrak{c}$ is real-valued measurable, and there is a Σ^2_1-well-ordering of $\mathbb{R}$?*

The idea is to use the large cardinal assumption to produce, given a Universally Baire set A, A-closed and sufficiently "fine structure-like" inner models of AD^+ over which forcing with $\mathbb{Q}_{\max}$ produces ZFC-models with a distinguished measurable cardinal. The measurable is used to produce a further extension, by forcing as in the $L[\mu]$ case. This provides us, combined with the fine structural features of the ground model, with an appropriate covering argument that can be used to correctly identify enough of HOD of the ground model to obtain the desired Σ^2_1-definition. The ground model can in fact be chosen so the forcing extension itself is A-closed, and this gives the result.

The rest of this talk is devoted to a sketch of Theorem 39.

Outline: By the Basis theorem, it suffices to show that if A is Δ^2_1 in $L(B, \mathbb{R})$ for some Universally Baire set B, then there is an A-closed model $M \models \mathsf{RVM}(\mathfrak{c}) + \mathsf{There\ is\ a}\ \Sigma^2_1\text{-}\mathsf{well\text{-}ordering\ of}\ \mathbb{R}$.

In order to apply Theorem 47 as stated, we find it necessary to work with a sequence of Universally Baire sets as opposed to just one.

The existence of this sequence is granted by Our Base Theory: Since B is Universally Baire, $L(B,\mathbb{R}) \models \mathsf{AD}^+$.

Fix $A_0 = A$. A is Δ^2_1 in $L(B,\mathbb{R})$. We obtain a sequence $\langle A_n \rangle_{n<\omega}$ and $B^* \subseteq \mathbb{R}$ such that

1. B^* is Universally Baire.
2. $\langle A_i \rangle_{i<\omega}$ is Δ^2_1 in $L(B^*,\mathbb{R})$.
3. $A_i^\sharp$ is Δ^2_1 in $L(A_{i+1},\mathbb{R})$.
4. $A_i^\sharp$ is $\Delta^1_1(A_{i+1})$.

To obtain the sequence, apply Lemma 50 inside $L(B,\mathbb{R})$ to obtain $A^* \in L(B,\mathbb{R})$ such that $A_0 \times A^*$ is $\Delta^2_1(L(A_0 \times A^*,\mathbb{R}))$, and take $A_1 = (A_0 \times A^*)^\sharp$ and $A_{n+1} = A_n^\sharp$ for $n \geq 1$.

We now proceed to sketch a high level description of the argument:

Let Γ be a pointclass. By $L_\Gamma(Y)[X]$ we mean the smallest transitive proper class P such that

1. $P \models \mathsf{ZF}$.
2. $Y \in P$.
3. $X \cap P \in P$.
4. P is A-closed for all A in Γ.

Theorem 56. *Assume Our Base Theory. Let ν be a normal measure in V on some cardinal κ. Then $N := L_{\langle A_i \rangle_{i<\omega}}(\mathbb{R})[\nu]$ exists and $N \models \mathsf{AD}^+ + \Theta = \theta_0$.* $\triangle$

We obtain this theorem as a corollary of the following two results:

Theorem 57. *Assume there are* ω^2+1 *Woodin cardinals. Suppose* A *is Universally Baire. Let* $\mathcal{F}$ *be the club filter on* $\mathcal{P}_{\omega_1}(\mathbb{R})$*. Then* $L(A,\mathbb{R})[\mathcal{F}] \models \mathsf{AD}^+$*, and* $\mathcal{F} \cap L(A,\mathbb{R})[\mathcal{F}]$ *is a measure in* $L(A,\mathbb{R})[\mathcal{F}]$*.* $\triangle$

Here, by $L(X_0,\dots,X_n)[Y]$ we mean the smallest transitive proper class model P of ZF such that $X_0,\dots,X_n \in P$ and $Y \cap P \in P$.

Theorem 58. $N \cap \mathcal{P}(\mathbb{R}) \subseteq L(\langle A_i\rangle_{i<\omega}, \mathbb{R})[\mathcal{F}]$*, and therefore* $N \models \mathsf{AD}^+$*.* $\triangle$

From Theorem 58 it follows that N is a sufficiently fine structural analogue in the context of AD^+ of $L[\mu]$. This allows us to show that it satisfies an appropriate version of covering.

Let $P = N[G_{\mathbb{Q}_{\max}}]$. Then $P \models \mathsf{ZFC} + \mathsf{GCH}$, and $\mu \cap N$ lifts to a normal measure witnessing measurability of κ in P. (Choice requires a little argument.)

Our model M will be of the form $M = P[G]$ where G is generic for the forcing at κ described in the section on $L[\mu]$. Notice that all the subsets of ω_1 in $M \setminus N$ are added by G, so

$$\mathcal{P}(\omega_1)^M = \mathcal{P}(\omega_1)^{P[G_{\mathtt{Random}_\kappa}]}.$$

Lemma 59. *Suppose $\mathcal{I}$ is an ω_1-dense ideal on ω_1 and that λ is such that whenever G is $\mathcal{P}(\omega_1)/\mathcal{I}$-generic over V and $j : V \to M$ is the ultrapower embedding in $V[G]$ generated by G, then $j(\lambda) = \lambda$. Then in $V^{\mathtt{Random}_\lambda}$,*

$$\mathcal{P}(\omega_1)/\mathrm{NS}_{\omega_1} \cong \mathtt{Coll}(\omega, \omega_1) \times \mathtt{Random}_\lambda. \quad \triangle$$

Since $j(\kappa) = \kappa$, where j is any generic embedding generated by NS_{ω_1}, the equality stated before Lemma 59 implies that in M, $\mathcal{P}(\omega_1)/\mathrm{NS}_{\omega_1} \cong \mathtt{Coll}(\omega, \omega_1) \times \mathtt{Random}_\kappa$, so we can apply Theorem 47.

The whole argument lays in the verification of enough Σ^2_1-facts about M. We can write $M = N[G_{\mathbb{Q}_{\max}}][G_{\mathtt{Random}_\kappa}][g]$, where g is the part of the generic which is actually coding the well-ordering. Suppose we have a possible second candidate M^*, so $M^* = N^*[G^*_{\mathbb{Q}_{\max}}][G^*_{\mathtt{Random}_{\kappa^*}}][g^*]$ for some N^* and some generic objects for the corresponding notions of forcing. All the correctness we can guarantee before hand in a Σ^2_1-way is that M^* is closed under ω-sequences, so from now on suppose that such is the case. We need to argue that the well-orderings coded in M and M^* coincide.

N^* involves a possibly fake sequence $\langle A_i^* \rangle_{i<\omega}$. What we show is that in fact $\langle A_i^* \rangle_{i<\omega} = \langle A_i \rangle_{i<\omega}$, and then we lift this agreement to show that HOD^N and HOD^{N^*} are sufficiently close. Once this is done, it follows that the well-orderings are indeed the same, and the argument concludes as in the case of $L[\mu]$.

We require a covering argument.

Theorem 60. *Every set of size ω_1 in M^* is covered by a set of size ω_1 in N^*.* $\triangle$

The first application of this result is the following:

Claim 61. $M^* \models A_i^*$ is ω_1-Universally Baire. $\triangle$

Since each A_i is ω_1-Universally Baire, it follows from the claim and 47 that A_i and A_i^* are Wadge compatible, and from this it is straightforward to arrange that they are in fact *equal*.

This shows that the possibly fake model N^* uses the same term relations as the true model N.

Now we argue that HOD^N is very nicely canonical, and this allows us to identify a significant part of HOD^N and HOD^{N^*}. Specifically, we show that HOD^N consists of a canonical construction built over HOD_Θ, $\Theta = \omega_3^M$.

For this, we rearrange $N = L_{\langle A_i \rangle_{i<\omega}}(\mathbb{R})[\nu]$ as $L_{\langle A_i \rangle_{i<\omega}}(Z)[\nu](\mathbb{R})$, a symmetric extension of $L_{\langle A_i \rangle_{i<\omega}}(Z)[\nu]$, where Z is a set of ordinals such that

$$L[Z] = \mathsf{HOD}^{L(\mathcal{P}(\mathbb{R}))^N}.$$

Recall that $Z \subseteq \Theta^{L(\mathcal{P}(\mathbb{R}))^N} = \omega_3^M$.

We get control over HOD from the fact that $Z = Z^*$, a consequence of covering.

Once this is arranged, we are able to identify a structure $\bar{N}$ such that

1. $N \subseteq \bar{N} \subseteq M$,
2. $\mathbb{R} \subset \bar{N}$, and
3. There is an elementary embedding $\pi : N \to \bar{N}$ fixing the ordinals.

Similarly, we find such an $\bar{N}^*$ between N^* and M^*, so $\mathbb{R} \subseteq \bar{N} \cap \bar{N}^*$, $\bar{N} \models \mathsf{AD}^+$ and $\bar{N}^* \models \mathsf{AD}^+$.

But then $\bar{N} = \bar{N}^*$, and this suffices for the comparison arguments and concludes the proof.

That $\bar{N} = \bar{N}^*$ follows from Theorem 58. Namely, if they are comparable (i.e., if their $\mathcal{P}(\mathbb{R})$ are comparable under Wadge reducibility) then they are equal, as they are definable in terms of a first order minimality condition that refers to predicates that coincide for both models. If they are not comparable, then their common part gives a model of $\mathsf{AD}_{\mathbb{R}}$, but the minimality condition that defines N, N^* does not grant enough room for this to happen, namely, no inner model P_1 of an AD^+ model P_2 such that $\mathbb{R} \subseteq P_1$ and $P_2 = L(A, \mathbb{R})[\mathcal{F}]$ can satisfy $\mathsf{AD}_{\mathbb{R}}$.

The identification of $\bar{N}$ is actually not so difficult.

Simplifying a bit, let $\bar{N} = N(\mathbb{R}^{G_{\mathbb{Q}\max} \times G_{\mathtt{Random}_\kappa}})$. In M, there is an embedding $\pi : N \to N_g$ which is the identity on the ordinals: π sends $\mathbb{R}^N$ to $\mathbb{R}^{N_g}$, ν to its lifting ν^{N_g}, and the term relations to themselves. It is easy to see that, as expected, this map is well-defined and elementary.

Define $\bar{N}^*$ similarly, and the sketch is complete.
$\square$

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