Goodstein's function

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Abstract

Goodstein's function $\mathcal{G} : \mathbb{N} \to \mathbb{N}$ is an example of a *fast growing* recursive function. Introduced in 1944 by R. L. Goodstein [9], Kirby and Paris [11] showed in 1982, using model theoretic techniques, that Goodstein's result that $\mathcal{G}$ is *total*, i.e., that $\mathcal{G}(n)$ is defined for all $n \in \mathbb{N}$, is not a theorem of first order Peano Arithmetic. We compute Goodstein's function in terms of the Löb-Wainer fast growing hierarchy of functions; from this and standard proof theoretic results about this hierarchy, the Kirby-Paris result follows immediately. We also compute the functions of the Hardy hierarchy in terms of the Löb-Wainer functions, which allows us to provide a new proof of a similar result, due to Cichon [2].
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Resumen

La función de Goodstein $\mathcal{G} : \mathbb{N} \to \mathbb{N}$ es un ejemplo de una función recursiva *de crecimiento rápido*. Introducida en 1944 por R. L. Goodstein [9], Kirby y Paris [11] demostraron en 1982, usando técnicas de teoría de modelos, que el resultado de Goodstein de que $\mathcal{G}$ es *total*, es decir, que $\mathcal{G}(n)$ está definida para todo $n \in \mathbb{N}$, no es un teorema de la Aritmética de Peano de primer orden. Calculamos la función de Goodstein en términos de la jerarquía de funciones de crecimiento rápido de Löb y Wainer; usando esto y resultados clásicos de teoría de la demostración acerca de esta jerarquía, el teorema de Kirby y Paris se sigue de inmediato. También calculamos las funciones de la jerarquía de Hardy en términos de las funciones de Löb y Wainer, con lo que obtenemos una nueva demostración de un resultado similar, debido a Cichon [2].

1 Introduction

Goodstein sequences were introduced in 1944 by Rueben Louis Goodstein [9], who proved that every such sequence is eventually zero. In 1982, Kirby and Paris [11] used model theoretic techniques (the method of *indicators*) to show that Goodstein's result, a first order statement in the language of arithmetic, is nevertheless not a theorem of first order Peano Arithmetic PA. Goodstein's function $\mathcal{G} : \mathbb{N} \to \mathbb{N}$ assigns to each n the first m such that the Goodstein sequence corresponding to n becomes zero from m on. In this paper we present a computation of Goodstein's function in terms of a classical "fast growing" hierarchy of functions due to Löb and Wainer, see Theorem 1.11. This is a well studied hierarchy, and the Kirby-Paris result is an immediate corollary of our computation and standard proof theoretic results about this hierarchy. A similar proof of the Kirby-Paris result was obtained by Cichon [2] using a different hierarchy originally introduced by Hardy. It is straightforward to compute the functions of the Hardy hierarchy in terms of the Löb-Wainer functions, and this calculation and Theorem 1.11 provide us with a new proof of Cichon's theorem, see Corollary 1.16.

This paper is organized as follows: In Subsection 1.1 we describe Goodstein's function, recall the definition of the Löb-Wainer hierarchy and the proof theoretic results (due to Wainer [16]) that we need, and state our main result, Theorem 1.11. In Subsection 1.2 we recall the definition of the Hardy hierarchy and derive Cichon's result from Theorem 1.11. Section 2 is devoted to the proof of Theorem 1.11; it is perhaps interesting to note that the argument organizes itself in a natural way as a transfinite induction of length ϵ_0. Finally, in Section 3 we briefly mention how the results of Kirby and Paris [11] follow from Theorem 1.11.

We want to thank William Sladek, whose interest in Goodstein's theorem and its unprovability in PA led to this paper.

1.1 Goodstein's theorem

Goodstein's theorem [9] provides a nice example of a finitary combinatorial result that cannot be proven without an explicit appeal to infinite sets, see Kirby and Paris [11]. This claim requires some explanation.

We assume acquaintance with the basic theory of ordinal numbers; the reader may find an introduction in almost any textbook in logic or set theory, like Cori and Lascar [3] or Kunen [12]. Recall that ϵ_0 is the first ordinal α such that $\alpha = \omega^\alpha$ (ordinal exponentiation).

For the reader not familiar with Peano Arithmetic PA or formal logic, it suffices to say any natural number theoretic statement can be easily expressed in the language of PA, and that PA is an appropriate formalization of the intuitive concept of "finitary mathematics", thus showing that PA cannot prove a statement $\mathcal{S}$ means that it is unavoidable to invoke infinite objects in any proof of $\mathcal{S}$.

We can make the above more precise in two ways: First, that PA captures finitary mathematics can be argued as follows: Recall that ZFC is the standard list of axioms for set theory, in which all of classical mathematics can be easily formalized, and is the accepted framework for carrying out such a formalization. Let $\mathsf{ZFC}_{\mathsf{fin}}$ be the theory that results when the axiom of infinite ("there are infinite sets") is removed from ZFC and replaced with its negation ("every set is finite", i.e., every set is in bijection with a natural number; formally this is stated as saying that there are no limit ordinals). Then it is easy to see that PA is bi-interpretable with $\mathsf{ZFC}_{\mathsf{fin}}$, which means that both theories are exactly the same, only stated in slightly different languages. Precisely: One can define recursive translations t and t' between the language of arithmetic and the language of set theory so that if ϕ is a theorem of PA, then its translation ϕ^t is a theorem of $\mathsf{ZFC}_{\mathsf{fin}}$ and, conversely, if ψ is a theorem of $\mathsf{ZFC}_{\mathsf{fin}}$, then $\psi^{t'}$ is a theorem of PA. Moreover, for any sentence ϕ in the language of arithmetic, PA proves that ϕ is equivalent to $(\phi^t)^{t'}$, and similarly for statements in the language of set theory and $\mathsf{ZFC}_{\mathsf{fin}}$. This argument provably comes from Ackermann [1]. A different justification of PA as the appropriate formalization of finitary mathematics can be found in the works of Gentzen, see [7] and [8], where the connection between PA and the ordinal ϵ_0 is highlighted.

Second, PA is sufficiently powerful to appropriately code and discuss *some* infinite sets; for example, any ordinal below ϵ_0 is formalizable inside PA, meaning in particular that if $\alpha < \epsilon_0$ and an arithmetic statement can be proven by finitary means together with an appeal to transfinite induction of length α, then the statement can be proven in PA. Precisely: The subsystem ACA_0 of second order arithmetic is conservative over PA for arithmetic sentences, but can refer to and discuss infinite objects. The infinite sets that can be appropriately discussed in ACA_0 are usually called *predicative*. That predicativism as understood by Weyl [17] is captured by ACA_0 follows from work of Feferman, see [5] and [6]. That ACA_0 is conservative over PA means that PA follows from ACA_0 and any arithmetic statement provable in ACA_0 (perhaps by explicit appeal to infinite objects) can also be derived purely within PA. For a discussion of ACA_0 and related theories, Simpson's monograph [14] is highly recommended.

Thus, if one shows that a statement $\mathcal{S}$ is not provable from PA, it follows that any proof of $\mathcal{S}$ must make explicit use of infinite, in fact, impredicative objects. Goodstein's theorem is an example of one such statement. It states that $\mathcal{G}(n)$ is defined for all n, where $\mathcal{G}$, Goodstein's function, is the number of steps that a certain process takes with input n before it halts. To describe the process, we need a couple of definitions.

Definition 1.1. The *depth-1* base b representation of $n \in \mathbb{N}$ is just the usual base b representation of n:

$$n = b^{m_1} n_1 + \cdots + b^{m_k} n_k$$

where $m_1 > \cdots > m_k \geq 0$ and $1 \leq n_i < b$ for each i.

By replacing each m_i with their base b representation we obtain the *depth-2* representation of m. In general, the *depth-*$(m+1)$ representation is obtained by replacing each m_i with their depth-m base b representation (so we iterate taking base b representations $m+1$ times).

For example, the depth-1 base 2 representation of 266 is $2^8 + 2^3 + 2^1$, its depth-2 base 2 representation is $2^{2^3} + 2^{2^1+1} + 2^1$ and so its depth-3 (or higher) base 2 representation is

$$266 = 2^{2^{2+1}} + 2^{2+1} + 2.$$

As with 266, notice that for any n and b, as m increases, the depth-$(m+1)$ base b representations of n eventually stabilize. (It is something of a tradition to mention 266 when discussing Goodstein's theorem, after one of the examples highlighted in Kirby-Paris [11].)

Definition 1.2. We call this stable representation the *complete base b representation* of $n \in \mathbb{N}$ (this is sometimes called the *super base b representation* of n).

Definition 1.3. The *change of base function* $R_b : \mathbb{N} \to \mathbb{N}$ takes a natural number n, and then replaces every b with $b+1$ in the complete base b representation of n.

Thus,

$$R_2(266) = 3^{3^{3+1}} + 3^{3+1} + 3 = 443426488243037769948249630619149892887.$$

Definition 1.4. The *Goodstein Sequence* beginning with n, $(n)_k$, is defined by $(n)_1 = n$ and for $k \geq 1$, $(n)_{k+1} = \begin{cases} R_{k+1}((n)_k) - 1 & \text{if} \quad (n)_k > 0 \\ 0 & \text{if} \quad (n)_k = 0. \end{cases}$

For example, the sequence for $n = 3$ is $3, 3, 3, 2, 1, 0, 0, \ldots$

Definition 1.5. The *Goodstein Function* $\mathcal{G} : \mathbb{N} \to \mathbb{N}$ is defined to be the smallest number k for which $(n)_k = 0$.

The main result of Goodstein [9] is the following:

Theorem 1.6. *The function $\mathcal{G}$ is well defined, i.e., $\mathcal{G}(n)$ exists for all n.*

Here are the first few values of the function $\mathcal{G}$:

$$\begin{aligned} \mathcal{G}(0) &= 1 \\ \mathcal{G}(1) &= 2 \\ \mathcal{G}(2) &= 4 \\ \mathcal{G}(3) &= 6 \\ \mathcal{G}(4) &= 3 \cdot 2^{402653211} - 2 \approx 6.895 \times 10^{121210694}. \end{aligned}$$

Contrast $\mathcal{G}(4)$ with the number of elementary particles in the universe, which is estimated[1] to be below 10^{90}; the number of *digits* of $\mathcal{G}(5)$ is much larger than

[1] See for example http://www.cs.umass.edu/∼immerman/stanford/universe.html

$\mathcal{G}(4)$. $\mathcal{G}$ is clearly a recursive function (i.e., there is a finite algorithm that from input n allows us to compute $\mathcal{G}(n)$); however, the values of $\mathcal{G}$ grow incredibly fast, so fast that $\mathcal{G}$ in fact eventually dominates any recursive function that PA can prove is defined for all inputs. This was originally proved by Kirby and Paris [11].

We prove Theorem 1.6 by presenting an "explicit" formula for $\mathcal{G}(n)$. It is as explicit as it is reasonable to expect; it describes $\mathcal{G}(n)$ in terms of the functions f_α of the fast growing hierarchy.

Any ordinal $\alpha < \epsilon_0$ can be written in a unique way as $\alpha = \omega^\beta(\gamma + 1)$ where $\beta < \alpha$. By transfinite recursion, define for limit $\alpha < \epsilon_0$ an increasing sequence $d(\alpha, n)$ cofinal in α by setting

$$d(\alpha, n) = \omega^\beta\gamma + \begin{cases} \omega^\delta n & \text{if } \beta = \delta + 1, \\ \omega^{d(\beta,n)} & \text{if } \beta \text{ is limit.} \end{cases}$$

The fast growing hierarchy $(f_\alpha)_{\alpha<\epsilon_0}$ of functions $f : \mathbb{N} \to \mathbb{N}$, due to Löb and Wainer [13], can now be defined as follows:

Definition 1.7.

1. $f_0(n) = n + 1$.
2. For $\alpha < \epsilon_0$, $f_{\alpha+1}(n) = f_\alpha^n(n)$, where the superindex indicates that f_α is iterated n times.
3. For limit $\alpha < \epsilon_0$, $f_\alpha(n) = f_{d(\alpha,n)}(n)$.

For example: $f_1(n) = 2n$, $f_2(n) = n2^n$ and $f_3(n)$ is (significantly) larger than a stack of powers of two of length n; $f_\omega(n) = f_n(n)$ grows like (the diagonal of) Ackermann's function, and $f_{\omega^{\omega^3}2}(n) = f_{\omega^{\omega^3}+\omega^{\omega^2 n}}(n)$, which itself requires some amount of time and effort to be computed. We caution the reader not to confuse the n^{th} iterate $f^n(m)$ of a function f applied to m with the n^{th} (multiplicative) power $f(m)^n$ of the number $f(m)$.

For $f, g : \mathbb{N} \to \mathbb{N}$, say that f is *eventually dominated* by g iff for all but finitely many values of n, $f(n) < g(n)$. Proofs of the following statements can be found in Wainer [16]:

Fact 1.8.

1. *Each f_α is strictly increasing.*
2. *If $\alpha < \beta < \epsilon_0$ then f_α is eventually dominated by f_β.*
3. *Each f_α is recursive, and provably total in* PA. □

Let $\zeta_0 = 0$ and $\zeta_{k+1} = \omega^{\zeta_k}$. The following is the main result of Wainer [16]:

Theorem 1.9. *If f is a recursive function, provably total in $I\Sigma_{k+1}$ (Peano Arithmetic with the induction axiom restricted to Σ_{k+1}-formulas), then f is eventually dominated by some f_α, $\alpha < \zeta_{k+1}$. In particular, any recursive f provably total in* PA *is eventually dominated by some f_α, $\alpha < \epsilon_0$.* □

Definition 1.10. Let $R_n^\omega(m)$ be the "change of base" function replacing each n by ω in the complete base n representation of m; for this, we express m as a *decreasing* sum of powers of n, so the resulting ordinal $R_n^\omega(m)$ is written in its Cantor normal form.

For example, since $266 = 3^{3+2} + 3^2 \cdot 2 + 3 + 2$, then

$$R_3^\omega(266) = \omega^{\omega+2} + \omega^2 2 + \omega + 2.$$

We can now state the main result of this paper, describing Goodstein's function in terms of the functions f_α.

Theorem 1.11. *1. Let*

$$n = 2^{m_1} + 2^{m_2} + \cdots + 2^{m_k}$$

where $m_1 > m_2 > \cdots > m_k$. *Let* $\alpha_i = R_2^\omega(m_i)$. *Then*

$$\mathcal{G}(n) = f_{\alpha_1}(f_{\alpha_2}(\ldots(f_{\alpha_k}(3))\ldots)) - 2.$$

2. More generally, let $\mathcal{G}_b(n)$ *be defined as* $\mathcal{G}$, *but we start by writing* n *in base* b *rather than 2, so (if not* 0*)* $(n)_2 = R_b(n) - 1$, $(n)_3 = R_{b+1}((n)_2) - 1$, *etc. Let*

$$n = b^{m_1} n_1 + \cdots + b^{m_k} n_k$$

where $m_1 > \cdots > m_k$ *and* $1 \leq n_k < b$ *be the base* b *representation of* n. *Let* $\alpha_i = R_b^\omega(m_i)$. *Then*

$$\mathcal{G}_b(n) = f_{\alpha_1}^{n_1}(f_{\alpha_2}^{n_2}(\ldots(f_{\alpha_k}^{n_k}(b+1))\ldots) - b.$$

For example, $\mathcal{G}(266) = f_{\omega^{\omega+1}}(f_{\omega+1}(6)) - 2$, because $266 = 2^{2^{2+1}} + 2^{2+1} + 2^1$ and $f_1(3) = 6$. Similarly,

$$\mathcal{G}(4) = f_\omega(3) - 2 = f_3(3) - 2 = 3 \cdot 2^3 \cdot 2^{3 \cdot 2^3} \cdot 2^{3 \cdot 2^3 \cdot 2^{3 \cdot 2^3}} - 2.$$

Goodstein's Theorem 1.6 follows at once from Theorem 1.11. As mentioned above, Theorem 1.11 also gives immediately as corollaries the unprovability results of Kirby and Paris [11], see Section 3.

1.2 The Hardy hierarchy

Goodstein's sequences can be defined in at least two ways. As opposed to how we define them here, one can also define them by, at each step, subtracting one from the current number and then increasing the current base. Call $g(n)$ the corresponding Goodstein's function. The reader should have no problem showing that the following holds:

Fact 1.12. *1.* g *is total iff* $\mathcal{G}$ *is total.*

2. *Assume that g is total. For any $n \in \mathbb{N}$, $g(n+1) = \mathcal{G}(n) + 1$.* $\square$

Our computation of $\mathcal{G}$ in terms of a fast growing hierarchy is not the first one: Cichon [2] analyzed g and found a formula for it in terms of the Hardy hierarchy (called this way after being introduced by Hardy [10] in order to 'exhibit' a set of reals of size $\aleph_1$), see also the paper [4] by Fairtlough and Wainer.

The Hardy hierarchy of functions is defined as follows:

Definition 1.13. 1. $H_0(n) = n$.

2. For $\alpha < \epsilon_0$, $H_{\alpha+1}(n) = H_\alpha(n+1)$.

3. For limit $\alpha < \epsilon_0$, $H_\alpha(n) = H_{d(\alpha,n+1)}(n)$.

One can easily provide an explicit computation of the members of the Hardy hierarchy in terms of the functions f_α; the following is obtained directly from the definitions by a straightforward induction on α:

Theorem 1.14. *For $0 < \alpha < \epsilon_0$, let*

$$\alpha = \omega^{\beta_0} n_0 + \cdots + \omega^{\beta_k} n_k$$

be the Cantor normal form of α, so $\alpha > \beta_0 > \cdots > \beta_k$ and $n_i > 0$ for all i. Then

$$H_\alpha(n) = f_{\beta_0}^{n_0}(\ldots(f_{\beta_k}^{n_k}(n+1))\ldots) - 1. \quad \square$$

In particular,

$$H_{\omega^\alpha}(n) = f_\alpha(n+1) - 1 \tag{1}$$

and we have the following:

Corollary 1.15. *If $\alpha \geq \beta$ then $H_\alpha \circ H_\beta = H_{\alpha+\beta}$.* $\square$

Cichon's computation, Corollary 1.16 below, is an immediate consequence of Theorems 1.11 and 1.14.

Corollary 1.16 (Cichon [2])**.** *For all $n \in \mathbb{N}$, $g(n) = H_{R_2^\omega(n)}(1)$.* $\square$

Formula (1) is proven in Fairtlough and Wainer [4] (among other places). Actually, (1) is stated there in terms of a slightly different hierarchy $(F_\alpha)_{\alpha<\epsilon_0}$, where it takes the form $H_{\omega^\alpha} = F_\alpha$. This hierarchy is also used in the paper [15] by Ketonen and Solovay; a straightforward induction from the definition given in [4] establishes the identity

$$F_\alpha(n) = f_\alpha(n+1) - 1$$

so, in terms of the F_α, Theorem 1.14 takes the form

$$H_\alpha(n) = F_{\beta_0}^{n_0}(\ldots(F_{\beta_k}^{n_k}(n))\ldots)$$

for $\alpha, \beta_0, \ldots$ as above.

Of course, Corollary 1.15 can also be proven directly from the definition. We caution the reader that the result as stated in [4, Lemma 2.17] (that the identity holds for all α, β) is incorrect, since for β limit, the identity $d(\alpha+\beta, n) = \alpha + d(\beta, n)$ may fail (consider for example $\alpha = 1$ and $\beta = \omega$).

Using Corollary 1.15 one may recover Theorem 1.14 by first arguing by induction on α that $H_{\omega^\alpha} = F_\alpha$ and then considering the Cantor normal form of an arbitrary ordinal below ϵ_0. Assuming Cichon's Theorem 1.16, this gives a different proof of Theorem 1.11.

What differentiates our argument from Cichon's is the analysis at limit ordinals that leads to Lemma 2.8.

2 The proof

To prove Theorem 1.11 it is better to work in terms of $B_a(n)$, the first *base* for which we reach zero when we start the process with the complete base a representation of n. Clearly, $B_2(n) = \mathcal{G}(n) + 1$.

Lemma 2.1. *For any a and any m, $B_a(a^m - 1) = f_{R_a^\omega(m)}(a) - 1$.*

Example 2.2. Using Theorem 1.11 we can compute $\mathcal{G}(15)$ as

$$\mathcal{G}(15) = f_{\omega+1}(f_\omega(f_1(f_0(3)))) - 2,$$

while using Lemma 2.1 gives us

$$\mathcal{G}(15) = \mathcal{G}(16 - 1) = f_{\omega^\omega}(2) - 2.$$

It may be instructive to reconcile both expressions: Notice that $f_0(3) = 4$ and $f_1(4) = 8 = f_2(2) = f_\omega(2)$, so the first expression simplifies to

$$\mathcal{G}(15) = f_{\omega+1}(f_\omega^2(2)) - 2 = f_{\omega+1}^2(2) - 2 = f_{\omega+2}(2) - 2.$$

Finally, we use Definition 1.7 repeatedly to find

$$f_{\omega^\omega}(2) = f_{\omega^2}(2) = f_{\omega 2}(2) = f_{\omega+2}(2).$$

Assuming Lemma 2.1, Theorem 1.11 is immediate by induction on k. For example, to prove item (1), just notice that $(n)_2 = 3^{R_2(m_1)} + \cdots + 3^{R_2(m_k)} - 1$ and $R_3^\omega(R_2(m)) = R_2^\omega(m)$ for any m.

We now proceed to the proof of Lemma 2.1. This requires a transfinite induction of length ϵ_0.

Definition 2.3. For $\alpha < \epsilon_0$ we define *exponential polynomials* $p_\alpha(x)$ by induction: If $\alpha > 0$, let

$$\alpha = \omega^{\beta_0} n_0 + \cdots + \omega^{\beta_k} n_k$$

be the Cantor normal form of α, so $\alpha > \beta_0 > \cdots > \beta_k$ and $n_i > 0$ for all i.

Define $N(\alpha)$ to be the largest integer mentioned in the Cantor normal form of α, so $N(n) = n$ for $n < \omega$ and, inductively,

$$N(\alpha) = \max\{N(\beta_0), \ldots, N(\beta_k), n_0, \ldots, n_k\}.$$

Set

$$p_\alpha(x) = x^{p_{\beta_0}(x)} n_0 + \cdots + x^{p_{\beta_k}(x)} n_k,$$

where $p_n(x) = n$ for all $n \in \omega$.

Definition 2.3 obviously implies the following inequality and identity.

Lemma 2.4. *$N(R^\omega_a(m)) < a$ and $p_{R^\omega_a(m)}(a) = m$ for all a, m.* $\square$

By Lemma 2.4, Lemma 2.1 follows immediately from the following, to which we devote the rest of this section.

Lemma 2.5. *For all $\alpha < \epsilon_0$ and all $a \geq N(\alpha)$, $B_a(a^{p_\alpha(a)} - 1) = f_\alpha(a) - 1$.*

Proof. The proof is by induction on α. For $\alpha = 0$ the result is clear.

Assume the result for α, and argue for $\alpha + 1$: $p_{\alpha+1}(a) = p_\alpha(a) + 1$ so $a^{p_{\alpha+1}(a)} - 1 = a^{p_\alpha(a)}(a-1) + a^{p_\alpha(a)} - 1$ and the induction hypothesis gives that

$$B_a(a^{p_{\alpha+1}(a)} - 1) = B_{f_\alpha(a)-1}((f_\alpha(a) - 1)^{p_\alpha(f_\alpha(a)-1)}(a-1))$$

which, for $a > 1$, equals $B_{f_\alpha(a)}(f_\alpha(a)^{p_\alpha(f_\alpha(a))}(a-2) + f_\alpha(a)^{p_\alpha(f_\alpha(a))} - 1)$. A straightforward induction, the base case of which we just displayed, now shows that for $k \leq a - 1$,

$$B_a(a^{p_{\alpha+1}(a)} - 1) = B_{f^{k+1}_\alpha(a)-1}((f^{k+1}_\alpha(a) - 1)^{p_\alpha(f^{k+1}_\alpha(a)-1)}(a - 1 - k)),$$

so in particular for $k = a - 1$, $B_a(a^{p_{\alpha+1}(a)} - 1) = B_{f^a_\alpha(a)-1}(0) = f_{\alpha+1}(a) - 1$, as wanted.

To treat the limit case we need a preliminary definition, compare with Ketonen and Solovay [15].

Definition 2.6. Define $\alpha \underset{n}{\rightarrow} \beta$, for $\beta \leq \alpha < \epsilon_0$, iff there is a sequence $\alpha_0 \geq \alpha_1 \geq \cdots \geq \alpha_k$ where $\alpha_0 = \alpha$, $\alpha_k = \beta$ and for all $i < k$, either α_i is successor and $\alpha_{i+1} = \alpha_i$, or else α_i is limit and $\alpha_{i+1} = d(\alpha_i, n)$.

A straightforward induction using Definition 2.3 shows the following:

Lemma 2.7. *If $\alpha \underset{a}{\rightarrow} \beta$ then $N(\alpha) \geq N(\beta)$, $f_\alpha(a) = f_\beta(a)$ and, if $a \geq N(\alpha)$, then $p_\alpha(a) = p_\beta(a)$.* $\square$

Suppose now that α is limit and the result holds for all $\beta < \alpha$. As before, $a^{p_\alpha(a)} - 1 = a^{p_\alpha(a)-1}(a-1) + a^{p_\alpha(a)-1} - 1$. Let $\gamma = R^\omega_a(p_\alpha(a) - 1)$. Then

$$B_a(a^{p_\alpha(a)} - 1) = B_{f_\gamma(a)-1}((f_\gamma(a) - 1)^{p_\alpha(f_\gamma(a)-1)}(a-1)),$$

and induction shows that for $k \leq a-1$,

$$B_a(a^{p_\alpha(a)}-1) = B_{f_\gamma^{k+1}(a)-1}((f_\gamma^{k+1}(a)-1)^{p_\alpha(f_\gamma^{k+1}(a)-1)}(a-1-k)),$$

so in particular for $k = a-1$ we have

$$B_a(a^{p_\alpha(a)}-1) = f_\gamma^a(a) - 1 = f_{R_a^\omega(p_\alpha(a)-1)+1}(a) - 1.$$

Lemma 2.8. *For all nonzero $\alpha < \epsilon_0$ and all $a \geq N(\alpha)$, $\alpha \underset{a}{\rightarrow} R_a^\omega(p_\alpha(a)-1)+1$.*

By Lemma 2.8, $\alpha \underset{a}{\rightarrow} R_a^\omega(p_\alpha(a)-1)+1$. By Lemma 2.7,

$$f_\alpha(a) = f_{R_a^\omega(p_\alpha(a)-1)+1}(a),$$

and the result follows.

All that remains is to prove Lemma 2.8, to which we now turn.

Proof. Once again, the argument is by induction. If $\alpha = \beta+1$, in particular if $\alpha = 1$, then $p_\alpha(a) = p_\beta(a)+1$ and $R_a^\omega(p_\beta(a)) = \beta$, as long as $a > N(\beta)$, i.e., $a \geq N(\alpha)$. Thus, $R_a^\omega(p_\alpha(a)-1)+1 = \beta+1 = \alpha$ in this case, as wanted.

Now suppose that α is limit and the result holds below α. By induction, we may as well assume that $\alpha = \omega^\beta$ for some nonzero $\beta < \alpha$. In particular, $\beta \underset{a}{\rightarrow} R_a^\omega(p_\beta(a)-1)+1$ if $a \geq N(\alpha) \geq N(\beta)$.

We have $p_\alpha(a) - 1 = a^{p_\beta(a)} - 1 = a^{p_\beta(a)-1}(a-1) + a^{p_\beta(a)-1} - 1$ and

$$R_a^\omega(p_\alpha(a)-1)+1 = \omega^{R_a^\omega(p_\beta(a)-1)}(a-1) + R_a^\omega(a^{p_\beta(a)-1}-1)+1.$$

Let $\gamma = R_a^\omega(p_\beta(a)-1)$. Since $\beta < \alpha$, by the induction hypothesis $\beta \underset{a}{\rightarrow} \gamma+1$ (so $\gamma < \beta$ and $\omega^\beta \underset{a}{\rightarrow} \omega^{\gamma+1}$) and

$$\omega^\gamma \underset{a}{\rightarrow} R_a^\omega(a^{p_\gamma(a)}-1)+1.$$

Then

$$\omega^\beta \underset{a}{\rightarrow} \omega^{\gamma+1} \underset{a}{\rightarrow} \omega^\gamma a = \omega^\gamma(a-1) + \omega^\gamma \underset{a}{\rightarrow} \omega^\gamma(a-1) + R_a^\omega(a^{p_\gamma(a)}-1)+1.$$

Finally, by Lemma 2.4, $p_\gamma(a) = p_\beta(a)-1$. This completes the proof. □

3 $\mathcal{G}$ and PA

An easy combinatorial argument (considering "walks" from larger ordinals to smaller ones along the sequences $d(\alpha,n)$) shows that to prove that the sequence $(f_\alpha)_{\alpha<\epsilon_0}$ is strictly increasing in the eventual domination order, it suffices to show that if α is limit and $n < m$, then $f_{d(\alpha,n)}(k) < f_{d(\alpha,m)}(k)$ whenever $k > n, N(\alpha)$. This leads to considering the relation $\underset{n}{\rightarrow}$ and Theorem 1.11 can be seen as a result of this analysis. A similar analysis of eventual domination is in Ketonen and Solovay [15], although the details vary somewhat, since the argument is carried out in terms of the sequence $(F_\alpha)_{\alpha<\epsilon_0}$.

From Theorem 1.11 and Wainer's Theorem 1.9 it follows immediately that $\mathcal{G}$ is not provably total in PA, since it is easy to see that $\mathcal{G}$ eventually dominates each f_α. For $m \in \omega$ define the m-Goodstein sequence beginning with n as $(n)_k$ above, but now instead of complete base b representations use only depth-$(m+1)$ base b representations. The proof of Theorem 1.11 also gives at once that the resulting function $\mathcal{G}_m$ eventually dominates each f_α, $\alpha < \zeta_{m+1}$, and therefore $\mathcal{G}_m$ is not provably total in $I\Sigma_{m+1}$, although $\mathcal{G}_m$ has rate of growth comparable to that of $f_{\zeta_{m+1}}$ and so it *is* provably total in $I\Sigma_{m+2}$. Similarly, Theorem $1'$ of Kirby and Paris [11] follows at once from the argument of Theorem 1.11.

References

[1] W. Ackermann. *Die Widerspruchsfreiheit der allgemeinen Mengenlehre*, Mathematische Annalen **114** (1937), 305–315.

[2] E. Cichon. *A short proof of two recently discovered independence proofs using recursion theoretic methods*, Proceedings of the American Mathematical Society **87** (1983), 704–706.

[3] R. Cori, D. Lascar. **Mathematical Logic. A course with exercises. Part II**, Oxford University Press (2001).

[4] M. Fairtlough, S. Wainer. *Hierarchies of provably recursive functions*, in **Handbook of Proof Theory**, S. Buss, ed., Elsevier-North Holland (1998), 149–207.

[5] S. Feferman. *Systems of predicative analysis, I*, The Journal of Symbolic Logic **29 (1)** (1964), 1–30.

[6] S. Feferman. *Systems of predicative analysis, II*, The Journal of Symbolic Logic **33** (1968), 193–220.

[7] G. Gentzen. *Die Widerspruchsfreiheit der reinen Zahlentheorie*, Mathematische Annalen **112 (1)** (1936), 493–565.

[8] G. Gentzen. *Beweisbarkeit und Unbeweisbarkeit von Anfangsfällen der transfiniten Induktion in der reinen Zahlentheorie*, Mathematische Annalen **119** (1943), 140–161.

[9] R. Goodstein. *On the restricted ordinal theorem*, The Journal of Symbolic Logic **9 (2)** (1944), 33–41.

[10] G. Hardy. *A theorem concerning the infinite cardinal numbers*, Quarterly journal of Mathematics **35** (1904), 87–94.

[11] L. Kirby, J. Paris. *Accessible independence results for Peano arithmetic*, Bulletin London Mathematical Society **14** (1982), 285–293.

[12] K. Kunen. **Set Theory. An introduction to independence proofs**, Elsevier Science Publishers (1980).

[13] M. Löb, S. Wainer. *Hierarchies of number theoretic functions. I*, Arch. Math. Logik Grundlagenforsch **14** (1970), 39–51.

[14] S. Simpson. **Subsystems of second order arithmetic**, Springer (1999).

[15] J. Ketonen, R. Solovay. *Rapidly growing Ramsey functions*, The Annals of Mathematics **113 (2)** (1981), 267–314.

[16] S. Wainer. *A classification of the ordinal recursive functions*, Arch. Math. Logik Grundlagenforsch **13** (1970), 136–153.

[17] H. Weyl. **Das Kontinuum: Kritische Untersuchungen über die Grundlagen der Analysis**, Veit (1918), reprinted in H. Weyl, E. Landau, B. Riemann. **Das Kontinuum und andere Monographien**, Chelsea, 2nd edition (1973).