Well-orderings of the reals

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Abstract

Well-orderings of the reals are an important tool in the investigation of the set theoretic structure of the real line. On the other hand, they exhibit pathological properties from the point of view of analysis.

The results in this talk explore the tension resulting from expecting nice properties of naturally defined classes of sets of reals, and the existence of simply definable well-orderings.

I. Preliminaries

Recall:

Definition 1. *Let* $(X, <)$ *be a totally ordered set. Then* X *is* well-ordered *by* $<$ *(or,* $<$ *is a* well-ordering *of* X*) iff*

$$\forall Y \subseteq X(\emptyset \neq Y \rightarrow Y \models \exists y \forall z\,(z \neq y \rightarrow y < z)).$$

I.e., X *is well-ordered iff every non-empty subset of* X *has a first element.*

Definition 2. AC (The axiom of Choice)*:*

Every set can be well-ordered.

In this talk we will work within the framework of the usual system for set theory, ZFC. In particular, AC holds.

Any two well-ordered sets $(X, <)$ and $(Y, \prec)$ can be compared in the following sense:

Exactly one of the following three situations holds:

- There is an order-preserving bijection $f : X \xrightarrow{\sim} Y$.
- There is a $y \in Y$ and an order-preserving bijection $f : X \xrightarrow{\sim} \{ z \in Y : z \prec y \}$.
- There is an $x \in X$ and an order-preserving bijection $f : \{ w \in X : w < x \} \xrightarrow{\sim} Y$.

I.e., one of X, Y is (via a renaming of its elements) an initial segment of the other.

There is a canonical system of well-ordered sets, the *ordinals*. Every well-ordered set is isomorphic to one of them.

Definition 3. *α is an ordinal iff $(\alpha, \in)$ is well-ordered, and transitive: $\beta \in \gamma \in \alpha \to \beta \in \alpha$.*

Let α, β be ordinals. Say that $\alpha < \beta$ iff there is an order preserving bijection from α into a proper initial segment of β.

- $\alpha < \beta$ iff $\alpha \in \beta$.

- The first ordinal is $\emptyset$. We call it 0.
- Given an ordinal α, there is a first ordinal β such that $\alpha < \beta$, namely $\beta = \alpha \cup \{\alpha\}$. We call this ordinal $\alpha + 1$.
- Hence, we identify the first few ordinals (the finite ordinals) with the natural numbers:
  - $0 = \emptyset$,
  - $1 = \{\emptyset\} = \{0\}$,
  - $2 = \{\emptyset, \{\emptyset\}\} = \{0, 1\}, \ldots$

But the ordinals do not stop here.

- $\omega = \{0, 1, 2, \ldots\}$ is the first infinite ordinal.
- The first ordinal larger than ω is $\omega + 1$. Then we have
  $\omega{+}2, \omega{+}3, \ldots, \omega{+}\omega, \omega{+}\omega{+}1, \ldots, \omega{+}\omega{+}\omega, \ldots$
- There is no largest ordinal. Given any collection S of ordinals, the set

  $$\bigcup \{ \alpha + 1 : \alpha \in S \}$$

  is an ordinal larger than all the elements of S.
- Since every set can be well-ordered, there are arbitrarily large ordinals.

The reason why ordinals and well-orders are useful is because they allow us to carry out arguments and constructions by induction, *even* if these inductions take longer than ω many steps.

We talk of *transfinite induction* in this setting.

Definition 4.

- *If an ordinal has the form* $\alpha + 1$ *we say it is* successor.
- *If* $\alpha > 0$ *is not successor, we say it is* limit.

For example, ω and $\omega + \omega$ are limit ordinals.

Example 1. *Let* $\underset{\sim}{\Sigma}^0_1$ *denote the collection of open subsets of* $\mathbb{R}$*, and define*

- $\underset{\sim}{\Pi}^0_\alpha$ = *the collection of complements of sets in* $\underset{\sim}{\Sigma}^0_\alpha$.
- $\underset{\sim}{\Sigma}^0_\lambda = \bigcup_{\alpha<\lambda} \underset{\sim}{\Sigma}^0_\alpha$ *for* λ *limit.*
- $\underset{\sim}{\Sigma}^0_{\alpha+1} = \{\bigcup_n A_n : A_n \in \underset{\sim}{\Pi}^0_\alpha\}$.

Let $\mathcal{B} = \bigcup_\alpha \underset{\sim}{\Sigma}^0_\alpha = \bigcup_\alpha \underset{\sim}{\Pi}^0_\alpha$. *Then* $\mathcal{B}$ *is the collection of Borel subsets of* $\mathbb{R}$.

The *cardinal* of a set is the set theoretic measure of its size. As in the *Principia*, we can think of the cardinality of X as the class of all sets bijective to X. By Choice, there is a more canonical way of defining cardinals.

Definition 5. *Let X be a set. Let α be the smallest ordinal such that there is a bijection $f : X \to \alpha$. Then we call $\alpha = |X|$ the cardinality of X, and we say that α is a cardinal number.*

- All finite ordinals are cardinals.
- ω is the first infinite cardinal. We also write $\aleph_0$ for ω.
- All the ordinals

  $$\omega, \omega + 1, \ldots, \omega + \omega, \ldots, \underbrace{\omega + \omega + \ldots}_{\omega \text{ times}}, \ldots$$

  are countable.
- The first uncountable ordinal is $\omega_1 = \aleph_1$.

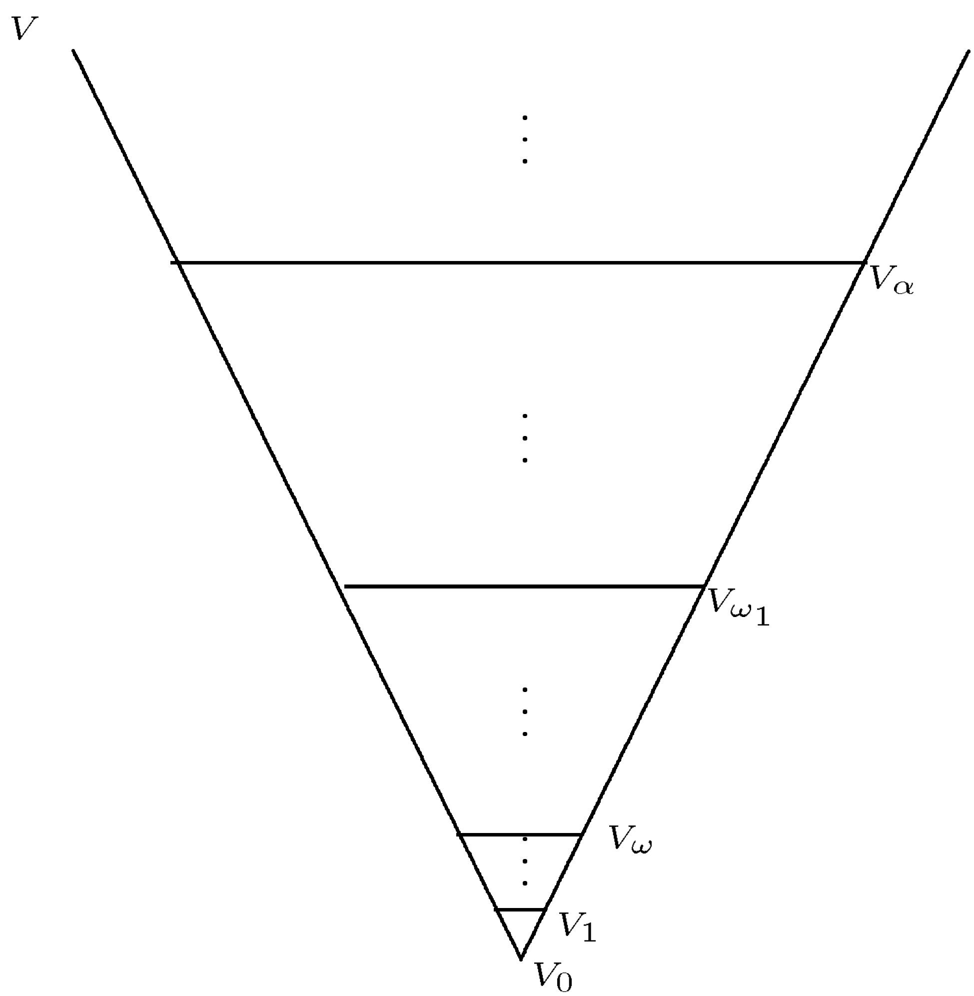
- (Cantor) Given any cardinal κ there is always a larger one, namely $2^\kappa = |\mathcal{P}(\kappa)|$.
- The smallest cardinal larger than κ is denoted by κ^+.
- The function $\alpha \mapsto \aleph_\alpha$ enumerates the infinite cardinals. Thus, $\aleph_0 = \omega$, $\aleph_1 = \omega^+$. Similarly, $\aleph_2 = \aleph_1^+$, $\aleph_3 = \aleph_2^+$, $\aleph_\omega = \sup\{\, \aleph_n : n < \omega \,\}$, etc.
- (CH) The *continuum hypothesis* is the assertion that $2^{\aleph_0} = \aleph_1$. Equivalently, for any infinite subset X of $\mathbb{R}$ there is a bijection $f : X \to \mathbb{N}$ or else there is a bijection $g : X \to \mathbb{R}$.

  We denote $2^{\aleph_0}$ by $\mathfrak{c}$.
- (GCH) For all infinite cardinals κ, $2^\kappa = \kappa^+$.

The standard picture of the set theoretic universe imagines the ordinals as given, and defines an increasing collection of sets inductively, as follows:

- $V_0 = \emptyset$.
- $V_{\alpha+1} = \mathcal{P}(V_\alpha)$.
- $V_\lambda = \bigcup_{\alpha<\lambda} V_\alpha$ for λ limit.

Then $V = \bigcup_\alpha V_\alpha$ is the universe of sets. Every set x belongs to some V_α. The least such α we call the rank of x, $\mathrm{rk}(x)$.


V
V_α
V_{ω_1}
V_ω
V_1
V_0

Definition 6. • *Let α be an ordinal. Its* cofinality *is the least β such that there is an increasing $f : \beta \to \alpha$ which is* cofinal, *i.e.,*

$$\forall\gamma \in \alpha\, \exists\delta \in \beta\, (\gamma \le f(\delta)).$$

We write $\mathrm{cf}\,(\alpha)$ *for its cofinality.*

- *An ordinal α is* regular *iff $\alpha = \mathrm{cf}\,(\alpha)$.*
- *Otherwise, it is* singular.

So, $\mathrm{cf}\,(\alpha + 1) = 1$. This is not interesting, and we only care about the cofinality of limit ordinals.

- If α is regular, then α is a cardinal.

- ω is regular.
- If κ is an infinite cardinal, κ^+ is regular.
- In particular, $\mathcal{B} = \bigcup_{\alpha<\aleph_1} \underset{\sim}{\Sigma}^0_\alpha$.
- $\mathrm{cf}\,(\aleph_\omega) = \omega$.

Definition 7. *A cardinal λ is* limit *iff it is $\aleph_\beta$ for some limit ordinal β. If it is of the form $\aleph_{\alpha+1}$ we say it is* succesor*.*

A regular limit cardinal is called weakly inaccessible*.*

Weakly inaccessibles are *big*. For example, if κ is inaccessible, then $\kappa = \aleph_\kappa$ and κ is the κ^{th}-cardinal λ such that $\lambda = \aleph_\lambda$.

II. Negative results

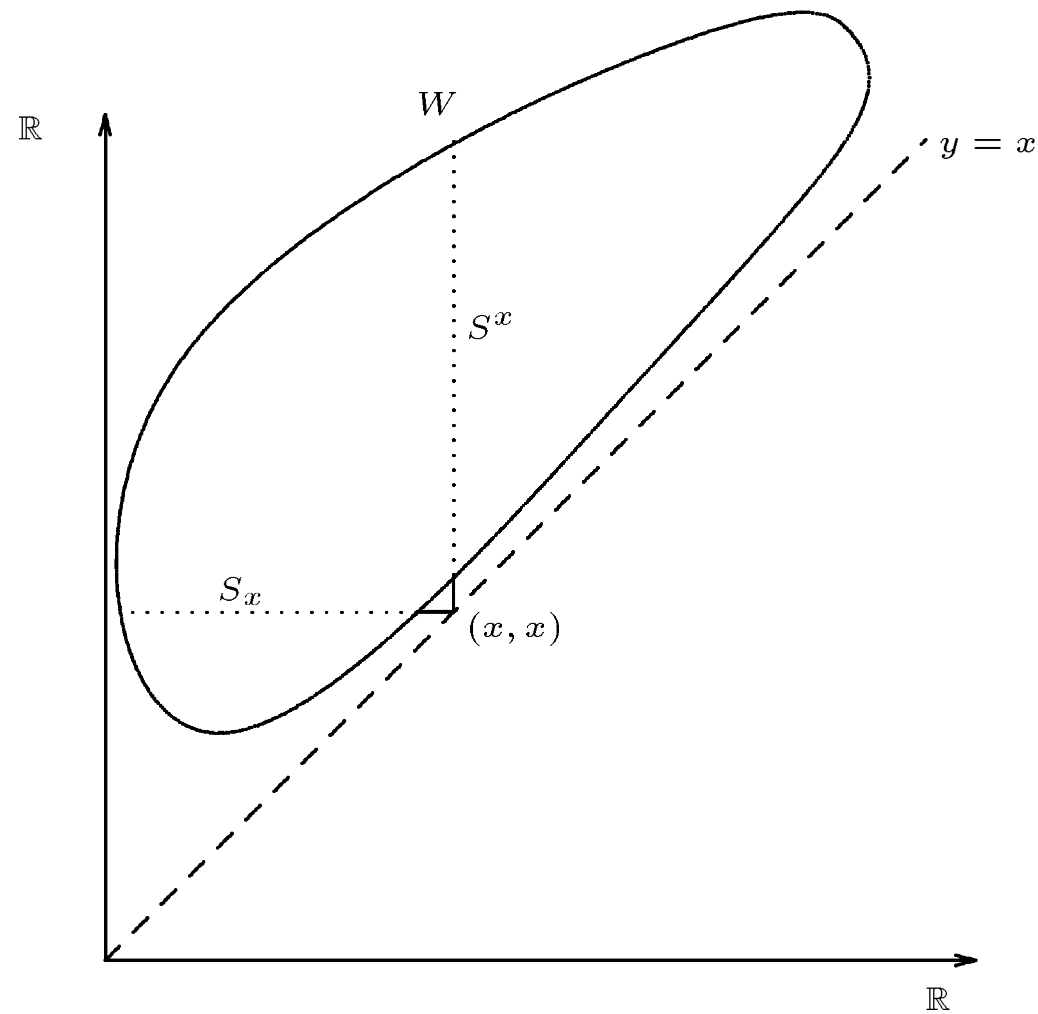
It is the working assumption of set theorists that sets of reals whose existence is solely granted by the Axiom of Choice are in general pathological, as opposed to those which can be 'explicitly' defined. A well-ordering of the reals (w.o.) is an example of such sets. Choice guarantees that the reals can be well-ordered, but no w.o. is Lebesgue measurable or has the Baire property.

Example 2 (Sierpinski). *Suppose* CH *holds. Under this assumption, we can arrange the reals on a list* $\langle\, x_\alpha : \alpha < \aleph_1 \,\rangle$. *The induced well-ordering*

$$W = \{\, (x_\alpha, x_\beta) : \alpha < \beta \,\}$$

is not Lebesgue measurable.

Remark. The result holds without any extra assumptions: We do not need CH, and the well-ordering does not need to be of o.t. $\mathfrak{c}$.



ℝ
W
y = x
S^x
S_x
(x, x)
ℝ

Proof For $\alpha < \omega_1$ let $S_\alpha = \{\, x_\beta : \beta < \alpha \,\}$ and $S^\alpha = \{\, x_\beta : \beta > \alpha \,\}$. Similarly, for $x \in \mathbb{R}$ define S_x as S_α, where $x = x_\alpha$, and do the same for S^x. Let $\mu(X)$ denote the Lebesgue measure of the set X. If W is measurable, it has measure 0, because

$$W = \int \mu(S_y)\, dy,$$

and each S_y, being countable, has measure zero.

On the other hand,

$$W = \int \mu(S^x)\, dx$$

has full measure, since each S^x has full measure. Contradiction. $\square$

A natural question to ask is how difficult is to define a w.o.

Of course, the answer depends superficially on what is meant by $\mathbb{R}$.

In set theory, we usually think of the reals mostly in terms of their coding potential, and therefore it is irrelevant whether $\mathbb{R}$ denotes

- The Euclidean reals.
- $\omega^\omega \cong \mathbb{R} \setminus \mathbb{Q}$.
- $2^\omega \cong$ The Cantor set.
- $[\omega]^\omega$, the infinite subsets of ω.

It turns out that the answer is independent of our choice.

All the spaces listed above are standard Borel spaces. The usual proof of the Cantor-Bendixon theorem actually shows that all of them have isomorphic Borel structures, the isomorphism given in each case by a Borel function. The isomorphism also identifies their natural continuous Borel measures.

II.a. Hierarchy of definability.

Define $\mathcal{B}(\omega^\omega)$, $\mathcal{B}(2^\omega)$, etc, exactly as for the case of $\mathbb{R}$. So $\underset{\sim}{\Sigma}^0_\alpha$ makes sense in any of these spaces.

- $\bigcup_n \underset{\sim}{\Sigma}^0_n$ is the class of arithmetic sets (in any of these spaces), i.e., letting $\mathbb{R}$ stand for any uncountable standard Borel spaces, sets of the form

$$\{\, x \in \mathbb{R} : \omega \models \varphi(x, y) \,\}$$

  where $y \in \mathbb{R}$ and φ is first order.

- All Borel sets are Lebesgue measurable.

So no w.o. can be defined arithmetically.

Definition 8. *(With $\mathbb{R}$ any uncountable standard Borel space) A set $X \subseteq \mathbb{R}$ is*

- $\undertilde{\Delta}^1_n$ *$(n > 0)$ iff it is* $\undertilde{\Sigma}^1_n$ *and* $\undertilde{\Pi}^1_n$.
- $\undertilde{\Pi}^1_n$ *iff its complement is* $\undertilde{\Sigma}^1_n$.
- $\undertilde{\Sigma}^1_1$ *iff it is the continuous image of a Borel set.*
- $\undertilde{\Sigma}^1_{n+1}$ *iff it is the continuous image of a* $\undertilde{\Pi}^1_n$ *set.*

Equivalently, X is $\undertilde{\Sigma}^1_n$ iff

$$X = \{\, y \in \mathbb{R} : \omega \models \underbrace{\exists y_1 \forall y_2 \ldots}_{n \text{ alternations}} \varphi(y_1, \ldots, y_n, y, z) \,\}$$

where $y_1, \ldots, y_n$ range over $\mathbb{R}$, $z \in \mathbb{R}$ and φ is first order.

Definition 9. *X is* projective *iff it is* $\undertilde{\Sigma}^1_n$ *for some n.*

The same definitions apply for subsets of $\mathbb{R}^k$, $k \geq 1$.

- $\mathcal{B} = \underset{\sim}{\Delta}^1_1$.
- Any $\underset{\sim}{\Sigma}^1_1$ set of reals is Lebesgue measurable, and therefore cannot be a w.o.

How difficult it is to define a w.o. is heavily dependent on the particular universe of sets one considers.

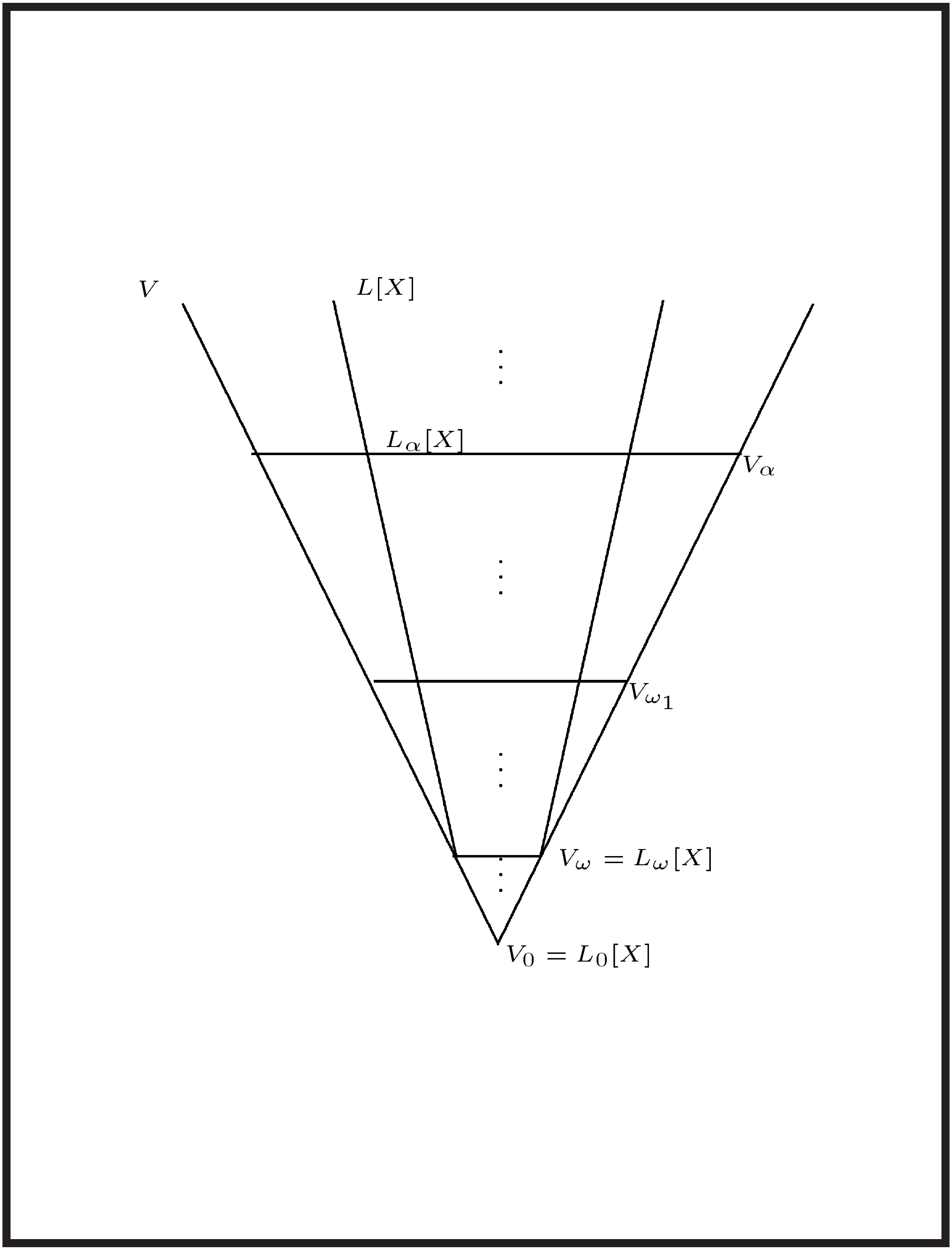
II.b. Inner Models

For example, in Gödel's constructible universe L, the reals admit a w.o. whose complexity is $\underset{\sim}{\Sigma}^1_2$.

On the other hand, if the reals admit a w.o. of such complexity, then there is a real x such that every real belongs to the universe constructible from x.

Definition 10. *Let X be a set. $L[X]$, the universe constructible from X, is defined by induction:*

- $L_0[X] = \emptyset$.
- $L_\lambda[X] = \bigcup_{\alpha<\lambda} L_\alpha[X]$ *for λ limit.*
- $L_{\alpha+1}[X] = \{\, y \subseteq L_\alpha[X] : y$ *is first order definable (from parameters) in* $(L_\alpha[X], \in, X \cap L_\alpha[X])\,\}$.
- $L[X] = \bigcup_\alpha L_\alpha[X]$.


V
$L[X]$
$L_\alpha[X]$
V_α
V_{ω_1}
$V_\omega = L_\omega[X]$
$V_0 = L_0[X]$

Superficially, V could be $L[X]$ for some X:

- Let $Y = X \cap L[X]$. Then
$$L[X] = L[Y] \models V = L[Y].$$
- $L[X] \models \mathsf{ZFC}$, the standard list of axioms for set theory.

This is superficial, because the list ZFC is incomplete; it does not include axioms that the set theoretic community now regards as true, but that imply, for example, that $V \neq L[X]$ for any X.

Gödel's constructible universe is $L = L[\emptyset]$.

Theorem 1 (Gödel). *If $V = L$, then there is a $\underset{\sim}{\Sigma}^1_2$ well-ordering of the reals. In fact, the well-ordering is Σ^1_2, i.e., no parameters are required.*

Proof (Sketch) 1. Suppose $x \in L$. Then there is α such that $x \in L_\alpha$ and $L_\alpha \models V = L$. Let $X \prec L_\alpha$ with $x \in X$ and X countable. Then $X \cong L_\beta$ for some countable β. Let $\pi : X \to L_\beta$ be the unique isomorphism. Then $\pi(x) = x$. Thus, $x \in L_\beta$.

(This shows $L \models \mathsf{CH}$.)

2. Fixing some ordering of formulas, let $x < y$, for $x, y \in L$ iff 'There are reals z_1, z_2 **coding well-orderings of** ω (in o.t. $\alpha \leq \beta$, respectively,) and there are reals y_1, y_2 such that y_1 codes L_α, y_2 codes L_β, $x \in L_\alpha$, $y \in L_\beta$, and if $\alpha = \beta$, then the formula defining x comes first than that defining y (and if the formulas are the same, then the parameters for x come first).' $\square$

II.c. Large Cardinals

A $\underset{\sim}{\Sigma}^1_2$ w.o. is ruled out as soon as

$$\forall x\, \exists y\, (y \notin L[x])$$

holds.

This is a consequence of the existence of large cardinals.

- The simplest example of a large cardinal is an inaccessible.

There is no formal definition of *large cardinal*, but there are certain generalities a large cardinal κ must satisfy. For example, we expect

- $V_\kappa \models \mathsf{ZFC}$ (Weakly inaccessibles have this property if GCH holds.)
- In fact, we expect κ to capture properties of V which cannot be expressed in a first order fashion.

The standard way of formalizing our last claim is via *reflection principles.* The standard way of generating reflection is via *elementary embeddings.*

Definition 11. *An elementary embedding* $j : V \to M$ *is a map between* $(V, \in)$ *and some proper class* (M, E) *such that* j *is elementary for formulas in the language of set theory.*

Definition 12. κ *is* measurable *iff there is an elementary embedding* $j : V \to M$ *with* M *transitive (i.e.,* $X \in Y \in M$ *implies* $X \in M$*, and* E *is just* $\in \restriction_{M \times M}$*) and with critical point* κ*.*

The *critical point* $\mathrm{cp}(j)$ of j is the first ordinal α such that $j(\alpha) \neq \alpha$. It is a consequence of choice that if $j : V \to M$ is elementary and M is transitive, then $\mathrm{cp}(j)$ exists, or else $j = \mathtt{id}$.

Large cardinals are usually defined by strengthening the requirement of measurability, for example, by asking M to be 'wide' enough to resemble V to some extent.

The recipe for getting reflection from these strengthenings is as follows:

> Suppose $j : V \to M$ is an elementary embedding with M transitive and critical point κ.
>
> Suppose κ has some property φ.
>
> Suppose M resembles enough of V so $M \models \varphi(\kappa)$.
>
> Then $M \models \exists\alpha < j(\kappa)\, \varphi(\alpha)$.
>
> By elementarity, $\exists\alpha < \kappa\, \varphi(\alpha)$.

I.e., we have *reflected* φ from κ to ordinals smaller than κ.

Warning. We should not expect the resemblance to be as strong as we wish. $V = M$ is impossible.

Example 3. *Let* κ *be measurable, let* $j : V \to M$ *be a witness, and let* x *be a real. Then* $\kappa > \omega$ *because* ω *is definable. Hence,* $j(x) = x$.

Moreover, $j\restriction_{L[x]} : L[x] \to L[x]$. *Therefore,* $V \neq L[x]$. *In fact, from the existence of* j *a real* $x^\sharp$ *can be defined such that the set of true sentences of* $(L[x], \in, x)$ *is computable from* $x^\sharp$. *By Tarski's result on the undefinability of truth,* $x^\sharp \notin L[x]$. *Hence, for each real* x *there is a real* y *not in* $L[x]$. *Thus, there is no* $\underset{\sim}{\Sigma}^1_2$ *w.o. of* $\mathbb{R}$.

Even more is true: All $\underset{\sim}{\Sigma}^1_2$ sets are Lebesgue measurable.

II.d. Inner Models for Large Cardinals

Analogues of the $L[X]$ models have been defined that allow for the existence of cardinals with large cardinal properties much stronger than measurability.

It is expected that all large cardinals set theorists study will be eventually ‘captured’ by one of these inner models.

Rather than defining these models, we list some of their features.

- They resemble L. For example, they are minimal models of the large cardinal assumption under consideration.
- They satisfy GCH.
- They are finestructural.

Example 4. *The analogue of L for a measurable has the form $L[\mu]$, where μ codes the embedding witnessing measurability. Any other model $L[\mu']$ can be recovered from $L[\mu]$ by iterating the embedding.*

The finestructural requirements ensure that the argument we gave for the case of L generalizes so the reals of these models have a w.o. which is simply definable. Exactly how simple depends on the specific large cardinals the model was designed to capture.

- L is obtained as the 'limit' of its building blocks L_α. Oversimplifying, finestructural models are the limit of models of the form $L_\alpha[\vec{E}]$, where $\vec{E}$ codes a sequence of embeddings.
- A key step in the argument for L was that any model 'resembling' an L_α (e.g., the collapse of a Skolem hull) had to be an L_α. This is no longer the case, but any 2 models 'resembling' an $L_\alpha[\vec{E}]$ can be successfully compared.
- The problem of obtaining a simple w.o. in a finestructural model reduces then to giving a simple definition for the comparison process.

If the models do not capture significant large cardinals, the comparison is relatively simple. In fact, if the models do not contain *Woodin cardinals*, their natural w.o. is $\utilde{\Sigma}^1_3$.

On the other hand:

Theorem 2 (Martin, Steel). *1. Let M_n be the minimal finestructural model for n Woodin cardinals. Then the reals of M_n admit a $\utilde{\Sigma}^1_{n+3}$-w.o.*

2. If there are n Woodin cardinals with a measurable above, then all $\utilde{\Sigma}^1_{n+2}$-sets are Lebesgue measurable.

3. Therefore, there are no projective well-orderings if there are ω Woodin cardinals.

Definition 13. *Let X be a transitive set. $L(X)$ is defined exactly as $L[X]$, except that $L_0(X) = X$.*

$L(X)$ is a model of Choice iff X is well-orderable in $L(X)$. For example, $L(\mathbb{R})$ is a model of Choice iff $\mathbb{R}$ is well-orderable via a w.o. in $L(\mathbb{R})$.

$L(\mathbb{R})$ contains all projective sets, thus the existence of a w.o. in $L(\mathbb{R})$ is in the presence of large cardinals an alternative to the existence of a projective w.o.

Theorem 3 (Woodin). *If there are ω Woodins with a measurable above then $L(\mathbb{R})$ is not a model of Choice.*

If no projective set can be a w.o., our next target must be the $\underset{\sim}{\Sigma}^2_n$ sets.

Definition 14. *1. A set of reals is* $\underset{\sim}{\Pi}^2_n$ *iff its complement is* $\underset{\sim}{\Sigma}^2_n$.

2. A set X is $\underset{\sim}{\Sigma}^2_n$ *iff we can write*

$$X = \{\, y \in \mathbb{R} : \omega \models \underbrace{\exists X_1 \forall X_2 \ldots}_{n \text{ alternations}} \varphi(\vec{X}, x, y) \,\}$$

where the X_i are sets of reals, $x \in \mathbb{R}$, and φ is projective.

Remark. An alternative, equivalent definition, is obtained by looking at definability over H_{ω_1}.

Theorem 4 (Woodin). *Suppose there is a proper class of measurable Woodin cardinals. Then all* $\underset{\sim}{\Sigma}_1^2$ *sets are Lebesgue measurable, provided that* CH *holds.*

In fact, under the hypothesis of the theorem (and CH) a much stronger statement holds:

The $\underset{\sim}{\Sigma}_1^2$ sets are *determined*, and the Σ_1^2 theory of the reals is *generically invariant* with respect to extensions satisfying CH.

Until very recently this was the upper bound for provable negative results.

Definition 15. *Let* $\varphi(x_1, \ldots, x_n)$ *be a formula. Let* $A \subseteq \mathbb{R}$. *The* A-Neeman game *given by* φ *is as follows:*

Two players I *and* II *alternate playing digits* $a_\alpha \in 2$ *for* ω_1 *many moves, to form a set* $a \in 2^{\omega_1}$.

I *wins iff there is a closed unbounded set* $C \subseteq \omega_1$ *such that for all* $\alpha_1 < \cdots < \alpha_n$ *in* C,

$$(H_{\omega_1}, \in, a, A) \models \varphi(\alpha_1, \ldots, \alpha_n).$$

Here, H_{ω_1} *is the collection of all* X *such that* $\{X\} \cup X \cup \bigcup X \cup \bigcup \bigcup X \cup \ldots$ *is countable.*

The game is determined iff either player has a winning strategy.

Building on results of Neeman, Woodin recently announced:

Theorem 5. *Suppose there is a proper class of supercompact cardinals. Let* Γ^∞ *be the collection of all* $A \subseteq \mathbb{R}$ *such that* A *is universally Baire.*

- *Suppose for each* $A \in \Gamma^\infty$, $\mathsf{ZFC} \vdash_\Omega$ *"All* A*-Neeman Games are determined".*

Then the Σ^2_2 *theory of reals is generically invariant with respect to extensions satisfying* $\diamondsuit$.

As an immediate consequence, if the hypothesis of the theorem holds, then no w.o. can be $\underset{\sim}{\Sigma}^2_2$, in the presence of $\diamondsuit$.

III. Positive Results

If the universe lacks a significant large cardinal structure, then it is possible to enlarge it, by the technique of *forcing*, to obtain a projective w.o.

- If $\mathbb{R} = \mathbb{R}^{L[x]}$, the reals of $L[x]$ for some real x, then $\mathbb{R}$ admits a $\Sigma^1_2(x)$-w.o.

Even better:

Theorem 6 (Harrington). *Suppose* $\omega_1 = \omega_1^{L[x]}$ *for some real* x*. Then there is a forcing extension of the universe with a* $\undertilde{\Sigma}^1_3$*-w.o.*

We can improve Harrington's result in an optimal way:

Definition 16. *κ is* strong *iff for all X there is an elementary embedding $j : V \to M$ with M transitive,* $\mathrm{cp}(j) = \kappa$ *and $X \in M$.*

Theorem 7. *Suppose there is no inner model $M \subseteq V$ such that*

$$M \models \text{There are } \omega \text{ strong cardinals.}$$

Then there is a forcing extension with a projective w.o.

In fact, under the hypothesis of the theorem, the *core model K*, a finestructural model, can be defined. Let n be the number of strongs in K. Then the w.o. we obtain is $\underset{\sim}{\Sigma}^1_{n+3}$.

This result is optimal due to the following result:

- Let H be $\mathtt{Add}(\omega, \omega_1)$-generic over V. Then no w.o. of $\mathbb{R}^{V[H]}$ belongs to $L(\mathbb{R})^{V[H]}$.

Theorem 8 (Woodin). *Suppose there are n strong cardinals. Let λ be larger than all of them. Let G be $\mathtt{Coll}(\omega, < \lambda)$-generic over V. Then the Σ^1_{n+2} theory of the reals (with parameters from $\mathbb{R}^{V[G]}$) cannot be changed by set forcing. In particular, no extension of the universe $V[G]$ admits a $\underset{\sim}{\Sigma}^1_{n+2}$-w.o.*

On the other hand, if we content ourselves with looking at forcing extensions of inner models, then some improvements are possible (nice behavior of simpler projective pointclasses can be imposed). For example:

Theorem 9 (Friedman, Schindler). *Let $n > 0$ and let M^n be the minimal inner model with n strongs and an inaccessible above. Then there is a forcing extension of M^n in which all $\underset{\sim}{\Sigma}^1_{n+3}$ sets are Lebesgue measurable and there is a Σ^1_{n+5}-w.o.*

Besides large cardinal axioms, ZFC is usually extended via *forcing axioms*, which are intended to formalize the idea that V is *complete.* The most popular of these axioms in Martin's axiom, MA.

Theorem 10 (Harrington). *There is a forcing extension of L where* MA *holds and there is a* $\undertilde{\Sigma}^1_3$*-w.o.*

We can improve Harrington's result as well:

Theorem 11. *Let M be the minimum inner model with a strong. Then there is a forcing extension of M where the following hold:*

- $\mathsf{SPFA}(\mathfrak{c})$.
- *Woodin's* ψ_{AC}.
- *There is a* Σ^1_4*-w.o.*

This result cannot be improved much. For example, $\mathsf{MM}(\mathfrak{c})$ implies Projective Determinacy, PD, and therefore is incompatible with the existence of a projective w.o.

Without restricting the large cardinal structure of the universe, no w.o. in $L(\mathbb{R})$ is to be expected. With CH, no w.o. can be Σ^2_1.

Theorem 12 (Abraham, Shelah). *There is a forcing extension of the universe satisfying* CH *where the reals admit a* Σ^2_2*-w.o.*

It must be mentioned that $\diamondsuit$ fails hopelessly in the model of the theorem, due to the restrictions on Suslin trees that their coding technique requires. By Woodin's Σ^2_2-absoluteness theorem, this is to be expected.

If CH fails, the result can be improved:

Theorem 13 (Woodin). *Suppose there is a weakly compact cardinal* κ. *Then there is a forcing extension where* $\mathfrak{c} = \kappa$, MA*(*σ*-centered) holds, and there is a* Σ^2_1*-w.o.*

Theorem 14 (Abraham, Shelah). *Let* κ *be the first inaccessible. Then there is a forcing extension where* $\mathfrak{c} = \kappa$, MA *holds, and there is a* Σ^2_1*-w.o.*

Remark. The published version of this result claims that κ can be any inaccessible, but the argument given there requires κ to be the first.

At the cost of a small value of $\mathfrak{c}$, the large cardinals are superfluous:

Theorem 15 (Solovay). *There is a forcing extension where* $\mathfrak{c} = \aleph_2$, $\mathsf{MA}(\sigma\text{-centered})$ *holds, and there is a* Σ^2_1*-w.o.*

This can be improved:

Theorem 16 (Abraham, Shelah). *Solovay's result can be obtained with* MA.

Our last example follows a different direction:

Definition 17. $\mathfrak{c}$ *is* real valued measurable *iff there is a probability measure on* $\mathcal{P}([0,1])$ *extending Lebesgue measure, which is* $\mathfrak{c}$*-complete.*

Theorem 17. *If* $\mathfrak{c}$ *is* real valued measurable, *then*

1. $\mathfrak{c}$ *is weakly inaccessible, and the* $\mathfrak{c}^{th}$*-weakly inaccessible.*
2. MA_{ω_1} *fails.*
3. *There is no w.o. in* $L(\mathbb{R})$*.*

Part 3 of the theorem follows from a characterization in terms of elementary embeddings, due to Solovay.

Theorem 18. *Let κ be measurable. Then there is a forcing extension of the universe where $\mathfrak{c} = \kappa$ is real valued measurable and there is a Σ^2_2-w.o.*

If the universe is small, then the result can be improved:

Theorem 19 (Woodin). *If $V = L[\mu]$, the minimal model for a measurable, then there is an extension where $\mathfrak{c}$ is real valued measurable and there is a Σ^2_1-w.o.*

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