The extent of squares under MRP

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Outline

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   - $\square(\kappa)$
2. The Mapping Reflection Principle
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   - Consequences of MRP
   - Definition
3. Squares under MRP
   - PFA
   - MRP
   - $\neg\square_{\kappa,\omega}$
   - $\square_{\kappa,\omega_1}$

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I want to present some recent results on the theory of the Mapping Reflection Principle, MRP, due to Moore, Sharon, Magidor, and to joint work with Veličković.

Square Principles

The combinatorial principle Square was introduced by Jensen (1972); Magidor describes it as providing uniform witnesses to the singularity of all ordinals between κ and κ^+. Several variants have been considered by Schimmerling, Todorčević and others.

Jensen showed that $\square_\kappa$ holds in L for all (infinite) cardinals κ, and Schimmerling and Zeman extended this by showing that it holds in (Jensen style) fine structural models as long as κ is not subcompact. Solovay showed that $\square_\kappa$ fails for κ at least as large as a strongly compact cardinal, and Jensen noticed that Solovay's argument shows that $\square_\kappa$ fails if κ is subcompact (a notion introduced by Jensen). Todorčević showed that it fails for all $\kappa > \omega$ under PFA. ($\square_\omega$ is trivially true.)

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Definition (Jensen)

Let κ be a cardinal and let $1 \leq \lambda \leq \kappa$. A sequence $\vec{C} = (\mathcal{C}_\alpha \colon \alpha < \kappa^+)$ is a $\square_{\kappa,\lambda}$-sequence iff for all $\alpha < \kappa^+$,

1. For all $c \in \mathcal{C}_\alpha$, c is club in α and $\mathrm{ot}(c) \leq \kappa$.
2. $1 \leq |\mathcal{C}_\alpha| \leq \lambda$.
3. For all $c \in \mathcal{C}_\alpha$ and all limit points β of c, $c \cap \beta \in \mathcal{C}_\beta$.

$\square_{\kappa,\lambda}$ holds iff there exists a $\square_{\kappa,\lambda}$-sequence. $\square_\kappa$ is $\square_{\kappa,1}$.

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Let κ be a cardinal and let $1 \leq \lambda \leq \kappa$. A sequence $\vec{C} = (\mathcal{C}_\alpha : \alpha < \kappa)$ is a $\square(\kappa, \lambda)$-sequence iff for all $\alpha < \kappa$,

1. $\mathcal{C}_{\alpha+1} = \{\{\alpha\}\}$ and each $c \in \mathcal{C}_\alpha$ is club in α if α is a limit ordinal.
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$\square(\kappa, \lambda)$ holds iff there exists a $\square(\kappa, \lambda)$-sequence. $\square(\kappa)$ is $\square(\kappa, 1)$.

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Theorem

1. *(Moore)* BPFA *implies that* $2^{\aleph_0} = \aleph_2$ *and* $L(\mathcal{P}(\omega_1)) \models$ AC.
2. *(C.-Veličković) In fact,* BPFA *implies that there is a well-ordering of* $\mathcal{P}(\omega_1)$ *that is* Δ_1*-definable in a subset of* ω_1.
3. *(Moore)* PFA *implies that there is a five element basis for the uncountable linear orders.*
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Definition

Let θ be a regular cardinal, let X be uncountable, and let $M \prec H_\theta$ be countable such that $[X]^\omega \in M$. A subset Σ of $[X]^\omega$ is *M-stationary* iff for all $E \in M$ such that $E \subseteq [X]^\omega$ is club, $\Sigma \cap E \cap M \neq \emptyset$.

Recall that the Ellentuck topology on $[X]^\omega$ is obtained by declaring a set open iff it is the union of sets of the form

$$[x, N] = \{Y \in [X]^\omega : x \subset Y \subseteq N\}$$

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A set mapping Σ is open stationary iff there is an uncountable set X and a regular cardinal θ such that $[X]^\omega \in H_\theta$, $\mathrm{dom}(\Sigma)$ is a club in $[H_\theta]^\omega$ and $\Sigma(M) \subseteq [X]^\omega$ is Ellentuck open and M-stationary, for every M in the domain of Σ.

MRP is the statement that if Σ is an open stationary set mapping, then there is a continuous $\in$-chain (a reflecting sequence for Σ)

$$\vec{N} = (N_\xi : \xi < \omega_1)$$

of elements in the domain of Σ such that for all limit ordinals $\eta < \omega_1$ there is $\xi < \eta$ such that $N_\nu \cap X \in \Sigma(N_\eta)$ for all ν such that $\xi < \nu < \eta$.

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It is easy to see that the failure of $\square(\kappa^+)$ implies the failure of $\square_\kappa$.

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Theorem (C.-Veličković)

MRP *refutes* $\Box(\kappa, < \omega)$ *for* κ *regular larger than* ω_1*.*

A simpler version of this argument can be used to establish:

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We sketch the (new) proof of Sharon's theorem. It begins by trying to follow Moore's argument.

Suppose otherwise, so MRP holds and we can find a cardinal $\kappa > \omega$ such that $\square_{\kappa,\omega}$ holds as witnessed by $\vec{C} = (\mathcal{C}_\alpha : \alpha < \kappa^+)$. Enumerate each $\mathcal{C}_\alpha$ as

$$\mathcal{C}_\alpha = \{C_{\alpha,n} : n < \omega\}.$$

Let $(\ell_\alpha : \alpha < \omega_1)$ be a ladder sequence, i.e., $\ell_\alpha \subseteq \alpha$ is cofinal of order type ω or 1 depending on whether α is limit or not. Given $\beta < \alpha < \omega_1$, define the height of β in α as

$$\mathrm{ht}_\alpha(\beta) = |\ell_\alpha \cap \beta|,$$

and notice that this height is an integer, and that if α is limit and β increases to α then $\mathrm{ht}_\alpha(\beta)$ increases to ω.

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Given a countable set X, let $\alpha_X = \sup(X \cap \omega_1)$.

Let θ be regular and sufficiently large, and for all $M \prec H_\theta$ containing $\vec{C}$, $(I_\alpha : \alpha < \omega_1)$, and $[\kappa^+]^\omega$, define $\Sigma(M) =$

$$\{N \in [M \cap \kappa^+]^\omega : \ \mathrm{ht}_{\alpha_M}(\alpha_N) = n \Rightarrow \sup(N) \notin \bigcup_{i<n} C_{\sup(M\cap\kappa^+),i}\}.$$

Then, for all such $M \prec H_\theta$, $\Sigma(M)$ is clearly Ellentuck open.

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If it is also M-stationary, one can apply MRP and obtain a reflecting sequence $(N_\xi : \xi < \omega_1)$. Then the set

$$E = \{\sup(N_\xi \cap \kappa^+) : \xi < \omega_1\}$$

is club in its supremum β, has order type ω_1, and yet has no limit points in common with $\bigcap \mathcal{C}_\beta$, a contradiction.

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$$E = \{\sup(N_\xi \cap \kappa^+) \colon \xi < \omega_1\}$$

is club in its supremum β, has order type ω_1, and yet has no limit points in common with $\bigcap \mathcal{C}_\beta$, a contradiction.

$\Sigma(M) =$

$$\{N \in [M \cap \kappa^+]^\omega : \ \mathrm{ht}_{\alpha_M}(\alpha_N) = n \Rightarrow \sup(N) \notin \bigcup_{i<n} C_{\sup(M\cap\kappa^+),i}\}.$$

Hence, it suffices to check that $\Sigma(M)$ is M-stationary. For this, fix an algebra $F : [\kappa^+]^{<\omega} \to \kappa^+$ in M. We can find in M an ω-club of ordinals $\beta < \kappa^+$ closed under F and of cofinality ω and, for each such β, a countable subset X_β cofinal in β and closed under F.

Towards a contradiction, suppose that whenever $\beta \in M$, $X_\beta \notin \Sigma(M)$. We can fix in M an ordinal $\zeta < \omega_1$ and an unbounded set of β such that $\alpha_{X_\beta} = \zeta$.

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Let $n = \mathrm{ht}_{\alpha_M}(\zeta)$. Then, for any such β, $\sup(X_\beta) \in \bigcup_{i<n} C_{\sup(M\cap\kappa^+),i}$.

It follows that for some $i < n$, the set of β such that

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is unbounded in κ^+. This is clearly a contradiction, as the set on the right has order type at most κ.

(I am oversimplifying a little.)

The argument refuting $\Box(\kappa, <\omega)$ is a bit more delicate.

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Question

Does MRP *refute* $\square(\kappa,\omega)$*?*

Sharon has extended his result in two directions. Recall that $\mathfrak{b} \leq non(\mathcal{M})$.

Theorem (Sharon)

Assume MRP*. Then:*

1. *If* $\kappa > \omega$ *then* $\square_{\kappa,<\mathfrak{b}}$ *fails.*
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Found at http://www.mathematicalquilts.com/

The end.