Projective well-orderings of the reals and forcing axioms

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Introduction

Throughout this talk, by a *real* we mean an element of the Cantor set $2^\omega$ or, equivalently, a subset of $\omega$.

Recall the *projective hierarchy* of pointclasses $\Sigma^1_k, \Pi^1_k, \Delta^1_k$, with effective versions $\Sigma^1_k(r), \Pi^1_k(r), \Delta^1_k(r)$ for $r$ a real, etc.

Our starting point is Gödel’s result that if $\mathbb{R} \subseteq L[r]$, then there is a $\Sigma^1_2(r)$ well-ordering of the reals.

We are interested in obtaining models with rich combinatorics (forcing axioms) and a reasonably controlled behavior of the set of reals (projective well-orderings). Of course, such models must be built under appropriate *smallness* assumptions.
I

Martin’s axiom
The first result in this direction is Leo’s:

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LONG PROJECTIVE WELLORDERINGS

Leo HARRINGTON*
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Theorem (Harrington)

There is a forcing extension of $L$ that preserves $\omega_1$ and makes $\mathfrak{c}$ as large as desired, where Martin’s axiom holds and there is a $\Delta^1_3$ well-ordering of the reals.

Contrast this with the following well-known result of Mansfield:

Theorem (Mansfield)

If there is a $\Sigma^1_2(r)$ well-ordering of the reals, then $\mathbb{R} \subseteq L[r]$; in particular, CH holds.
This leads to several natural questions. For example:

1. Leo’s definition of the well-ordering requires the use of a parameter. Can we obtain a model with a lightface $\Delta^1_3$ well-ordering?
2. Is this result compatible with (small) large cardinals?
3. Can we replace MA with a stronger forcing axiom?
The first question was answered by Sy:
Theorem (Friedman)

There is a forcing extension of $L$ that preserves $\omega_1$, makes $\mathfrak{c} = \omega_2$, and where MA holds and there is a $\Delta^1_3$ well-ordering of the reals.

The argument uses class forcing techniques and goes by implementing David’s trick: One builds an iteration along which:

1. Generics are added to all “small” ccc forcings.
2. A well-ordering of the reals is coded by adding branches to certain Suslin trees.
3. This coding (by large uncountable objects) is localized in appropriate countable sets. For this, a reshaping predicate (of size $\omega_1$) is added. Countable transitive models containing traces of this predicate correctly decode fragments of the well-ordering.

At the end, MA holds, and almost disjoint coding “captures” the predicate by a real.
In the extension, the well-ordering is defined by claiming that $x < y$ iff there is a real “coding a predicate” such that in any countable transitive model of enough set theory containing the real, $x$ and $y$, we have that $x < y$ according to the model’s trees.
There are several problems trying to generalize this approach:

1. The argument uses in an essential way that at any stage the reshaping predicate has size $\omega_1$. This forces us to have $\mathfrak{c} = \omega_2$. Contrast this with Harrington’s result where the continuum can be made arbitrarily large.

2. David’s trick has no natural analogue in the presence of large cardinals. The key point is that “candidate” models are automatically correct; this traces back to condensation in $L$. Once iterations are required to carry out comparisons, this correctness fails.

3. Trying to obtain stronger forcing axioms with this approach, the usual proof that the axiom holds at the end of the iteration goes by a reflection argument. This reflection cannot be applied here since the tail-ends of the iteration are not set-forcings. This is not an issue in the proof above, since one only needs to consider ccc forcings of size at most $\omega_1$ to begin with.
There has been some recent work towards obtaining models of MA with large continuum and a $\Delta^1_3$ well-ordering of the reals. For example:

**Theorem (Fischer-Friedman-Zdomskyy)**

There are forcing extensions of $L$ that preserve $\omega_1$ while forcing $b = c = \omega_3$, and adding a $\Delta^1_3$ well-ordering.

More work is needed, though: We need to obtain larger values of $c$. Jensen coding techniques should allow one to do this for this theorem, but these methods do not seem to interact well with the iterations needed to obtain MA, and it is likely that one would need to solve first the problem of iterations with large continuum.
There are several additional obstacles when one wants to obtain models of MA with a projective well-ordering of the reals, in the presence of large cardinals:

1. (Welch) If every real has a sharp but there is no inner model with a strong cardinal, then a $\Sigma^1_3(r)$ well-ordering of the reals implies that $\mathbb{R} \subseteq K_r$; in particular, CH holds.

2. (Woodin) On the other hand, if there is a strong cardinal, there is a forcing extension where no further forcing extension admits a $\Sigma^1_3$ well-ordering of the reals. In general, if there are $n > 0$ strong cardinals below $\kappa$, then $\Sigma^1_{n+3}$ generic absoluteness holds after forcing with $\text{Coll}(\omega, 2^{2\kappa})$.

3. (Hjorth) If every real has a sharp and MA holds, then every $\Sigma^1_3$ set of reals is Lebesgue measurable.
II
The bounded proper forcing axiom
Trying to obtain models with stronger forcing axioms, the key insight was Woodin’s:
**Definition (Woodin)**

$\psi_{AC}$ is the statement that for any pair $S, T$ of stationary co-stationary subsets of $\omega_1$, there is an $\alpha < \omega_2$ such that whenever $G$ is $P(\omega_1)/\text{NS}_{\omega_1}$-generic,

$$S \in G \iff \alpha \in j(T)$$

where $j$ is the generic ultrapower embedding.

Given $S, T$, denote by $\alpha(S, T)$ the smallest ordinal as in the definition above above.
Lemma

Assume there is a measurable cardinal. Given $S, T$, the instance of $\psi_{AC}$ corresponding to $S$ and $T$ can be forced with semiproper forcing.

The (natural) forcing takes a measurable $\kappa$ and collapses it to $\omega_1$ by adding (using closed initial segments) a bijection $\pi : \omega_1 \rightarrow \kappa$ such that for a club $C \subseteq \omega_1$ we have

$$S \cap C = \{ \alpha \in C \mid \pi[\alpha] \in T \}.$$

Woodin proves in his book the weaker statement that this poset is stationary set preserving. He also shows that if MM holds, then so does $\psi_{AC}$.

In fact, we have that if, say,
- BSPFA, the bounded semiproper forcing axiom, holds, and there is a measurable, or
- BSPFA is forced by a standard iteration of length $\tau$ a limit of measurable cardinals,
then $\psi_{AC}$ holds.
Theorem (2003)

Assume $V = L[\mathcal{E}]$ is a Mitchell-Steel model with one strong cardinal, and there are no inner models with Woodin cardinals. Then the standard iteration for SPFA$(c)$ and BSPFA$^{++}$ gives a model where $\psi_{AC}$ holds and there is a $\Sigma^1_6$ well-ordering of the reals.

The point here is that we obtain the projective well-ordering "for free", without the need for additional "coding" stages in the forcing.
\(\psi_{AC}\) gives us a definable well-ordering: Partition \(\omega_1\) into countably many stationary sets \((S_n)_{n<\omega}\). Given \(r \in 2^\omega\), associate to it the set
\[
S_r = \bigcup_{n \in r} S_n
\]
and let \(\gamma_r = \alpha(S_r, S_0)\). Then
\[
x < y \iff \gamma x < \gamma y
\]
is a definable \((\Sigma_2)\) well-ordering of the reals.

The presence of sharps gives us \(\Sigma_3^1\) absoluteness between \(V\) and the extension. From this it follows that we can find a partition \((S_n \mid n < \omega)\) that is \(\Delta_3^1\) in the codes. This allows us to turn the well-ordering just described into a projective well-ordering, and its complexity turns out to be \(\Sigma_6^1\).
This result suggested that BPFA or a similar forcing axiom should suffice to ensure the existence of definable well-orderings. What we require is a coding device similar to $\psi_{AC}$ but provable from BPFA. The fundamental new idea is Justin’s:
Justin introduced the Mapping Reflection Principle MRP, and proved that PFA implies MRP and that MRP implies a $\psi_{AC}$-like principle (he calls it $\nu_{AC}$) from which we can obtain a $\Sigma_2$ well-ordering of the reals. Moreover, BPFA suffices to prove the instance of MRP required to prove $\nu_{AC}$.

In fact, $\nu_{AC}$ allows us to conclude the existence of projective well-orderings in, say, models of BPFA obtained by the standard forcing over $L$, but the complexity of these well-orderings is larger than $\Sigma_6^1$.

To improve this complexity seems to require an additional amount of absoluteness.
In the paper below, we obtain a definable well-ordering with the required amount of absoluteness:
Recall that \((C_\alpha \mid \alpha < \omega_1)\) is a \(C\)-sequence iff \(C_\alpha\) is a cofinal subset of \(\alpha\) of smallest possible order-type for all \(\alpha < \omega_1\).

**Theorem (Caicedo-Veličković)**

BPFA implies that for every \(C\)-sequence \(\vec{C}\) there is a \(\Sigma_1\) well-ordering of the reals in the parameter \(\vec{C}\).

As with the \(\nu_{AC}\)-well-ordering, the proof of this result factors through MRP.
The definability of the well-ordering we obtain has an (unexpected) consequence:

**Corollary**

*If BPFA holds and $\omega_1 = \omega^L_1$ then there is a $\Sigma^1_6$ well-ordering of the reals.*

Notice the difference with the previous results: Here we obtain a projective well-ordering directly from BPFA and a smallness assumption, while previously we needed some additional knowledge of how the model of BPFA was obtained. What remains is to see whether one can improve the projective complexity of the well-ordering.
Theorem (Caicedo-Friedman)

Assume BPFA.

1. If $\omega_1 = \omega_1^L$, then there is a $\Sigma^1_3$-well-ordering of the reals.

2. In fact, if $\omega_1$ is not remarkable in $L$, then there is a $\Sigma^1_3$-well-ordering.

3. Moreover, if there is no inner model with $\omega$ strong cardinals, and $\omega_1 = \omega_1^K$, then there is a projective well-ordering.

The complexity of the well-ordering in item 3 matches what is best possible modulo projective absoluteness. This uses a computation of the initial segments of $K$ due to Schindler.
The theorem follows from the Caicedo-Veličković result thanks to the following observation:

**Theorem (Caicedo-Friedman)**

Assume $\text{MA}_{\omega_1}$ and that $\omega_1 = \omega_1^{L[r]}$ for some real $r$. Let $R(x, y)$ be a $\Sigma_1$ relation on reals in parameter $\omega_1$. Then $R$ is $\Sigma_3^1(r)$.

The proof of this result resembles a David's trick argument but it is in fact simpler:
Assume MA\(_{\omega_1}\) and \(\omega_1 = \omega_1^L\), and suppose that \(R(x, y, \omega_1)\) holds. Since \(R\) is \(\Sigma_1\), there is a set \(A \subseteq \omega_1\) coding the membership relation of a transitive model of size \(\omega_1\) of ZFC\(^-\) + \(R(x, y, \omega_1)\). But then there is a \(\beta\) (of size \(\omega_1\)) such that \(A \in L_\beta[A, x, y]\), \(L_\beta[A, x, y] \models ZFC^-\), and

\[
L_\beta[A, x, y] \models \psi(x, y, A),
\]

where \(\psi\) is a statement indicating that the model coded by \(A\) satisfies \(R(x, y, \omega_1)\).

There is then a club \(C\) of ordinals \(\alpha < \omega_1\) and a sequence \((M_\alpha \mid \alpha \in C)\) of countable models such that

\[
\forall \alpha \in C \ (M_\alpha \prec L_\beta[A, x, y] \text{ and } M_\alpha \cap \omega_1 = \alpha).
\]
Let $Y \subseteq \omega_1$ code $(C, A)$ in the sense that $\text{Odd}(Y) = A$ and if $Y_0 = \text{Even}(Y)$ and $\{c_\alpha \mid \alpha < \omega_1\}$ is the increasing enumeration of the elements of $C$, then:

- $Y_0 \cap \omega$ codes a well-ordering of type $c_0$.
- $Y_0 \cap [\omega, c_0) = \emptyset$.
- For all $\alpha$, $Y_0 \cap [c_\alpha, c_\alpha + \omega)$ codes a well-ordering of type $c_{\alpha+1}$.
- For all $\alpha$, $Y_0 \cap [c_\alpha + \omega, c_{\alpha+1}) = 0$.

Note that the following statement (*) holds:

Whenever $\mathcal{M}$ is a countable transitive model of $\text{ZFC}^-$ such that $Y \cap \omega_1^\mathcal{M} \in \mathcal{M}$, $\omega_1^\mathcal{M} = (\omega_1^L)^\mathcal{M}$, and $x, y \in \mathcal{M}$, then

$$\mathcal{M} \models R(x, y, \omega_1^\mathcal{M}).$$

This is because for any such $\mathcal{M}$, $\delta = \omega_1^\mathcal{M}$ belongs to $C$, $A \cap \delta \in \mathcal{M}$ and codes a transitive model $N \in \mathcal{M}$. As $N$ satisfies $R(x, y, \omega_1^\mathcal{M})$ and $R(x, y, z)$ is a $\Sigma_1$ formula, so does $\mathcal{M}$. 

Caicedo
Projective well-orderings of the reals and forcing axioms
Let $\vec{r}$ be the canonical $L$-sequence of $\omega_1$ many almost disjoint reals, and let $\mathbb{P}$ be the almost disjoint coding forcing that codes $Y$ as a real $r$ relative to $\vec{r}$. The generic produces a subset $z$ of $\omega$ such that, for all countable $\beta$, $z$ is almost disjoint from $r_\beta$ exactly if $\beta$ belongs to $Y$. Then the following property (**) holds:

For any countable transitive model $M$ of ZFC without $L$ such that $z, x, y \in M$ and $M \models \omega_1 = \omega_1^L$, we have that $M \models R(x, y, \omega_1^M)$.

This is because any such $M$ can reconstruct $Y \cap \omega_1^M$ and so we can apply (*).

Since we are assuming MA$_{\omega_1}$, there is in $V$ a real $z$ as above. This shows that there is a $\Sigma^1_3$ statement $\varphi_R(x, y)$ (namely, the assertion that there is a real $z$ such that (**) holds) such that $\varphi(x, y)$ holds whenever $R(x, y, \omega_1)$ does.

Conversely, if $\varphi_R(x, y)$ holds as witnessed by the real $z$, then (**) holds without the restriction that $M$ be countable, by reflection. But then $R(x, y, \omega_1)$ holds.
We have proved that, under reasonable smallness assumptions, BPFA implies the existence of projective well-orderings of optimal complexity. A gap remains:

**Question**

If BPFA holds and $0^+$ does not exist, does it follow that there is a $\Sigma^1_3$-well-ordering of the reals?

And an embarrassing question:

**Question**

Is $\text{MA} + \omega_1 = \omega_1^L$ consistent with the nonexistence of a projective well-ordering of the reals?
The end.