What are Super-real Fields?

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I. Algebras of Continuous Functions

Definition 1 *An* algebra *is a (real or complex) vector space with a multiplication* $\cdot$ *(and possibly with extra structure)*

$$\mathcal{A} = (A, +, \cdot, \cdots)$$

such that

$$\begin{aligned} \mathcal{A} \models (\forall x, y, z) (\quad & x(yz) = (xy)z, \\ & x(y+z) = xy + xz, \\ & \text{and } (x+y)z = xz + yz \quad). \end{aligned}$$

Let $\leq$ be a partial ordering of A. Denote by A^+ or $\mathcal{A}^+$ the nonnegative elements of A:

$$A^+ = \{ a \in A : \mathcal{A} \models a \geq 0 \}.$$

We say that $\mathcal{A}$ is an *ordered algebra* if and only if

$$\begin{aligned} \mathcal{A} \models (\forall x, y, z)\,(x < y \rightarrow \quad & x + z < y + z \text{ and} \\ & z + x < z + y) \end{aligned}$$

and $ab \in \mathcal{A}^+$ whenever $a, b \in \mathcal{A}^+$.

An algebra $\mathcal{A}$ is *normable* if and only if there is a map $\|\cdot\| : \mathcal{A} \to \mathbb{R}^+$ such that for all $a, b \in \mathcal{A}$ and all scalars r,

1. $\|a\| \geq 0$, $\|a\| = 0$ only for $a = 0$;
2. $\|ra\| = |r|\|a\|$;
3. $\|a + b\| \leq \|a\| + \|b\|$; and
4. $\|ab\| \leq \|a\|\|b\|$.

It is *unital* if and only if there is an element $\mathbf{1} \in \mathcal{A}$ such that

$$\mathcal{A} \models (\forall x)\, x\mathbf{1} = \mathbf{1}x = x.$$

If $\mathcal{A}$ is normable, it can be assumed that $\|\mathbf{1}\| = 1$.

It is a *Banach algebra* if and only if, moreover, $(\mathcal{A}, \|\cdot\|)$ is complete.

All the algebras we consider are assumed to be commutative.

In this talk, we restrict our attention to a specific kind of algebras: The spaces $C(X)$ of continuous, and $C^b(X)$ of continuous and bounded, real-valued functions with domain a topological space X, and their quotients.

$C(X)$ and $C^b(X)$ are ordered algebras under the ordering

$$f \leq g \Leftrightarrow (\forall x \in X)\, f(x) \leq g(x).$$

The first result is that we can restrict our attention to the case where X is completely regular.

Theorem 1 *For any topological space X there is a completely regular space Y and a continuous surjection*

$$\tau : X \to Y$$

such that the map $g \mapsto g \circ \tau$ is an isomorphism of $C(Y)$ onto $C(X)$ and of $C^b(Y)$ onto $C^b(X)$.

Proof For $x, y \in X$, say $x \sim y$ if and only if $f(x) = f(y)$ for all $f \in C(X)$. Let

$$Y = X/\sim$$

and let τ be the quotient map. Turn Y into a completely regular topological space by endowing it with the *weak topology* induced by

$$C = \{ g \in {}^{Y}\mathbb{R} : g \circ \tau \in C(X) \},$$

that is, the smallest topology for which

$$C \subset C(Y).$$

This works. □

From now on, all our spaces are completely regular.

Let me recall some standard constructions:

Let $\mathcal{A}$ be a real or complex algebra. A *character* on $\mathcal{A}$ is a homomorphism from $\mathcal{A}$ onto $\mathbb{R}$ or $\mathbb{C}$, respectively.

Definition 2 $\Phi_{\mathcal{A}}$ *is the space of characters on A.*

If $\mathcal{A}$ is a complex Banach algebra, then $\Phi_{\mathcal{A}} \subset C(A, \mathbb{C})$ and $\|\varphi\| \leq 1$ for each $\varphi \in \Phi_{\mathcal{A}}$, so $\Phi_{\mathcal{A}}$ is a subset of the closed unit ball of $\mathcal{A}'$.

The weak-$*$ topology on $\mathcal{A}'$ makes $\Phi_{\mathcal{A}}$ a locally compact space. For $\mathcal{A}$ unital, it is in fact compact and nonempty.

Definition 3 *Let $\mathcal{A}$ be a unital complex Banach algebra. The* Gelfand transform *of $\mathcal{A}$ is the homomorphism* $\hat{}: \mathcal{A} \to C(\Phi_{\mathcal{A}}, \mathbb{C})$ *given by*

$$\hat{a}(\varphi) = \varphi(a) \quad \text{for all } \varphi \in \Phi_{\mathcal{A}}.$$

Let $\|\cdot\|_X$ be the sup norm. Then $(C^b(X, \mathbb{C}), \|\cdot\|_X)$ is a Banach algebra.

Notice that $\Phi_{C^b(X,\mathbb{C})} \neq \emptyset$; for example, let $x \in X$, and define φ_x by

$$\varphi_x(f) = f(x).$$

Then $\varphi_x \in \Phi_{C^b(X,\mathbb{C})}$.

Theorem 2 $\Phi_{C^b(X,\mathbb{C})} \cong \beta X$*, the Stone-Čech compactification of X. The Gelfand transform $f \mapsto \hat{f}$ is an isometric isomorphism between $C^b(X, \mathbb{C})$ and $C(\beta X, \mathbb{C})$.* □

II. Ideals and Filters

We are interested in quotients of $C(X)$ given by (proper) ideals. The most useful ideals for our purposes satisfy an extra condition defined in terms of zero sets.

Definition 4 *For* $f \in C(X)$, *the* zero set *of* f *is*

$$\mathsf{Z}_f(X) = \mathsf{Z}(f) = \{ x \in X : f(x) = 0 \}.$$

For I *an ideal in* $C(X)$, *let*

$$\mathsf{Z}[I] = \{ \mathsf{Z}(f) : f \in I \},$$

and write $\mathsf{Z}(X) = \mathsf{Z}[C(X)]$.

Example 1 *If* X *is a metric space, then* $\mathsf{Z}(X)$ *is the family of closed subsets of* X.

Definition 5 *A nonempty* $\mathcal{F} \subset \mathsf{Z}(X)$ *is a z-***filter** *if and only if:*

1. $\emptyset \notin \mathcal{F}$,
2. $F_1 \cap F_2 \in \mathcal{F}$ *for* $F_1, F_2 \in \mathcal{F}$, *and*
3. *if* $F \in \mathcal{F}$ *and* $F \subset G \in \mathsf{Z}(X)$, *then* $G \in \mathcal{F}$.

A z-filter $\mathcal{U}$ *maximal under inclusion is called a z-***ultrafilter.**

Example 2 *If* X *is discrete,* $C(X) = {}^X\mathbb{R}$ *and* $\mathsf{Z}(X) = \mathcal{P}(X)$, *so a z-filter is just a filter in the Boolean algebra* $(\mathcal{P}(X), \subseteq)$.

Definition 6 *For* $\mathcal{F}$ *a z-filter on* X, *set*

$$\mathsf{Z}^{-1}[\mathcal{F}] = \{ f \in C(X) : \mathsf{Z}(f) \in \mathcal{F} \}.$$

Theorem 3 *Let I be an ideal in $C(X)$ and $\mathcal{F}$ a z-filter on X. Then*

- $\mathsf{Z}[I]$ *is a z-filter.*
- $I \subset \mathsf{Z}^{-1}[\mathsf{Z}[I]]$.
- $\mathsf{Z}^{-1}[\mathcal{F}]$ *is an ideal.*
- $\mathcal{F} = \mathsf{Z}[\mathsf{Z}^{-1}[\mathcal{F}]]$. $\square$

Definition 7 *A z-ideal I is one such that*

$$I = \mathsf{Z}^{-1}[\mathsf{Z}[I]].$$

So, I is a z-ideal if and only if $g \in I$ whenever $\mathsf{Z}(f) = \mathsf{Z}(g)$ for some $f \in I$.

Not all ideals are z-ideals; for example, let $X = \mathbb{R}$ and $I = (\mathrm{id})$. Then

$$\mathsf{Z}^{-1}[\mathsf{Z}[I]] = \{ f \in C(\mathbb{R}) : f(0) = 0 \} \supsetneq I.$$

Theorem 4 *Let I be an ideal in $C(X)$ and $\mathcal{F}$ a z-filter on X. Then*

- $\mathsf{Z}^{-1}[\mathcal{F}]$ *is a z-ideal.*
- $\mathsf{Z}^{-1}[\mathsf{Z}[I]]$ *is the minimum z-ideal containing I.* $\square$

Any maximal ideal $\mathcal{M}$ is a z-ideal.

Theorem 5 *Let $\mathcal{M}$ be a maximal ideal and $\mathcal{U}$ a z-ultrafilter. Then*

- $\mathsf{Z}[\mathcal{M}]$ *is a z-ultrafilter.*
- $\mathsf{Z}^{-1}[\mathcal{U}]$ *is a maximal ideal.* $\square$

For $Z \in \mathsf{Z}(X)$, let $\mathrm{cl}(Z)$ denote its closure inside βX. For any $Z_1, Z_2 \in \mathsf{Z}(X)$, it is the case that

$$\mathrm{cl}(Z_1 \cap Z_2) = \mathrm{cl}Z_1 \cap \mathrm{cl}Z_2.$$

Thus, if $\mathcal{U}$ is a z-ultrafilter on X,

$$\mathcal{G} = \{\, \mathrm{cl}Z : Z \in \mathcal{U} \,\}$$

has the finite intersection property. By maximality of $\mathcal{U}$,

$$\bigcap \mathcal{G} = \{\mathfrak{p}_{\mathcal{U}}\}$$

is a singleton.

For $\mathfrak{p} \in \beta X$,

$$\mathcal{U}_{\mathfrak{p}} = \{\, Z \in \mathsf{Z}(X) : \mathfrak{p} \in \mathrm{cl}Z \,\}$$

is a z-ultrafilter on X.

Theorem 6 *(Gelfand-Kolmogorov) The correspondence $\mathcal{U} \leftrightarrow \mathfrak{p}$ is a bijection, and so the points of βX correspond to the maximal ideals of $C(X)$.* □

III. Quotients

Let $\mathcal{A}$ be an algebra; I an ideal in $\mathcal{A}$; and $S \subset \mathcal{A}$ a non-empty set, disjoint from I, and closed under multiplication. Then there is a prime ideal Q with $I \subset Q$ and $Q \cap S = \emptyset$.

Thus, the intersection of all the prime ideals extending I is the set of elements of which some power belongs to I.

Prime ideals are particularly useful, because if P is prime in $\mathcal{A}$, then $\mathcal{A}/P$ is an integral domain.

In the case $\mathcal{A} = C(X)$, we can say a great deal about its prime ideals. For example:

Theorem 7 *Let P be a prime ideal in $C(X)$.*

1. *For any $f \in C(X)$, either $f \in f^+ + P$, or $f \in f^- + P$.*
2. *The prime ideals containing P are totally ordered by inclusion.*
3. *Any z-ideal containing P is a prime ideal.*
4. *There is a unique maximal ideal containing P.*

Proof We give a proof of *1.* and *3.*:

1. $f = f^+ + f^-$, but $f^+ f^- = 0 \in P$.
3. Suppose $I \supset P$ is a z-ideal. Since $\mathsf{Z}(f^n) = \mathsf{Z}(f)$ for all f and $n \in \mathbb{N}$, I is the intersection of all the prime ideals containing it. But by *2.*, this set is a chain. Hence, I is prime. □

Definition 8 *Let P be prime in $C(X)$. Then*

$$\mathcal{A}_P = C(X)/P.$$

Let π_P be the quotient map. For $a, b \in \mathcal{A}_P$, say that $a \geq b$ if and only if there are $f, g \in C(X)$ with $f - g \in C(X)^+$, $a = \pi_P(f)$ and $b = \pi_P(b)$.

$\mathcal{A}_P$ so defined, is a commutative, unital algebra. $\leq$ is a total ordering of $\mathcal{A}_P$, because since $\pi_P(f) \in \{\pi_P(f^+), \pi_P(f^-)\}$ for any f, then either $\pi_P(f) \geq 0$ or $\pi_P(f) \leq 0$.

Clearly, $(\mathcal{A}_P, \leq)$ is an ordered algebra.

In the case P is actually maximal, much more can be said:

If $\mathcal{M}$ is a maximal ideal in $C(X)$, then $\mathcal{A}_{\mathcal{M}}$ is an integral domain with no non-trivial ideals. Thus, it is a field. We identify $\mathbb{R}$ with the obvious copy of it inside $\mathcal{A}_{\mathcal{M}}$.

Definition 9 *Let K be a field properly extending $\mathbb{R}$. K is* hyper-real *if and only if K is isomorphic (via a map fixing $\mathbb{R}$) to some $\mathcal{A}_{\mathcal{M}}$ with $\mathcal{M}$ maximal in some $C(X)$.*

Example 3 *Let X be discrete, and let $\mathcal{M}$ be maximal in $C(X) = {}^X\mathbb{R}$. Let $\mathcal{U} = \mathsf{Z}[\mathcal{M}]$. Then $\mathcal{U}$ is an ultrafilter on X and*

$$\mathcal{A}_{\mathcal{M}} = \mathbb{R}^X/\mathcal{U}$$

is the ultrapower *of $\mathbb{R}$ by $\mathcal{U}$.*

Many features of this example hold in general:

Theorem 8 *Let $\mathcal{M}$ be maximal. Then $\mathcal{A}_{\mathcal{M}}$ is a real-closed field.* $\square$

In fact, Łoś's theorem holds:

Theorem 9 *Let $\mathcal{M}$ be maximal, and set $\mathcal{U} = \mathsf{Z}[\mathcal{M}]$. For any $f_1, \ldots, f_n \in \mathcal{A}_{\mathcal{M}}$ and any formula ϕ in the language of ordered rings,*

$$\mathcal{A}_{\mathcal{M}} \models \phi(\pi_{\mathcal{M}}(f_1), \ldots, \pi_{\mathcal{M}}(f_n)) \iff \{ x \in X : \mathbb{R} \models \phi(f_1(x), \ldots, f_n(x)) \} \in \mathcal{U}.$$

Proof For atomic formulas this is immediate from the definition. The inductive step for $\wedge$ and $\neg$ are clear.

Since $\mathcal{A}_{\mathcal{M}}$ is a real-closed field, any formula ϕ is equivalent to a quantifier-free formula ϕ'. This implies the result. $\square$

Also, as in the case of ultrapowers, hyper-real fields are $\aleph_1$-saturated. We are mainly concerned with a consequence of saturation:

Definition 10 *Let* $(\mathbb{P}, \leq)$ *be a totally ordered set. Say that* $\mathbb{P}$ *is an* η_1-set *whenever, for all countable* $S_1, S_2 \subset \mathbb{P}$, *if*

$$(\forall x \in S_1)(\forall y \in S_2)\, \mathbb{P} \models x < y,$$

then there is some $s \in \mathbb{P}$ *such that*

$$(\forall x \in S_1)(\forall y \in S_2)\, \mathbb{P} \models x < s < y.$$

Theorem 10 *Let* K *be a hyper-real field. Then* K *is an* η_1*-set.* □

Theorem 11 [1] *Let* K *be a real-closed,* η_1*-field. Then*

$$|K| = |K|^{\aleph_0}. \quad \square$$

Theorem 12 [1] *Let* κ *be a cardinal such that* $\kappa^{\aleph_0} = \kappa$. *Then there is a hyper-real field* K *such that* $|K| = \kappa$. □

Corollary 1 [1] *Assume* GCH. *Then not every hyper-real field is an ultrapower.*

Proof Let $\kappa = \beth_{\omega_1}$. Then $\kappa^{\aleph_0} = \kappa$ and there are hyper-real fields of size κ.

Under GCH, results of Keisler and Prikry imply that for any ultrapower K of $\mathbb{R}$, if $|K| = \kappa > \mathfrak{c}$, then $\kappa^{\aleph_1} = \kappa$.

Since, by König's lemma, $\beth_{\omega_1}^{\aleph_1} > \beth_{\omega_1}$, the result follows. $\square$

Open question:

- (Without extra assumptions) are there hyper-real fields which are not ultrapowers?

Arguing as above, a strong positive answer would be a consequence of:

- Let κ be a strong limit cardinal of cofinality bigger than ω. Then no ultrapower of $\mathbb{R}$ has size κ.

Unfortunately, such a general statement does not hold. Shelah and Jin [6] have shown that is consistent to have counterexamples assuming the existence of supercompact cardinals.

It would suffice to show that no ultraproduct of $\mathbb{R}$ can have size κ for κ singular strong limit of cofinality ω_1. But even the case $\kappa = \beth_{\omega_1}$ is still open.

Theorem 13 *(CH) Let $\mathcal{U}$ and $\mathcal{V}$ be nonprincipal ultrafilters on* $\mathbb{N}$*. Then*

$$\mathbb{R}^{\mathbb{N}}/\mathcal{U} \cong \mathbb{R}^{\mathbb{N}}/\mathcal{V}.$$

Proof Being hyper-real fields, both fields are real-closed η_1-sets. Since $|\mathbb{R}^{\mathbb{N}}| = \mathfrak{c}$, they both have size $\mathfrak{c}$.

Let K be a real-closed field with transcendence basis over $\mathbb{R}$ of size at most $\aleph_1$. Say $\{ a_\sigma : \sigma < \omega_1 \}$ is such a basis. Define inductively

- $K_0 = \mathbb{R}$.
- $K_{\alpha+1} \subset K$ is the real-closure of $K_\alpha(a_\alpha)$, i.e., $K_{\alpha+1}$ is real-closed and algebraic over $K_\alpha(a_\alpha)$.
- $K_\lambda = \bigcup\{ K_\alpha : \alpha < \lambda \}$ for λ limit.

Thus, $K = \bigcup_\alpha K_\alpha$.

A (somewhat careful) back-and-forth argument using such a representation for $K = \mathbb{R}^{\mathbb{N}}/\mathcal{U}$ and $K = \mathbb{R}^{\mathbb{N}}/\mathcal{V}$ completes the proof, once we show that every hyper-real field has transcendence degree over $\mathbb{R}$ at least $\mathfrak{c}$.

But this follows easily: Let $R \subset \mathbb{R}$ be a vector space basis for $\mathbb{R}$ over $\mathbb{Q}$. Let K be a hyper-real field, and let $u \in K$ be infinitely large. Then

$$\{ u^r : r \in R \}$$

is algebraically independent over $\mathbb{R}$; in effect, if

$$p(X_1, \ldots, X_k) = \sum_{s=1}^{m} a_s X_1^{n_{s,1}} \ldots X_k^{n_{s,k}}$$

is a nonzero polynomial in $\mathbb{R}[X_1, \ldots, X_k]$, then $|p(u^{r_1}, \ldots, u^{r_k})|$ is infinitely large whenever $r_1, \ldots, r_k \in R$ are distinct. $\square$

IV. Super-real fields

Now suppose P is a prime ideal in $C(X)$, but not necessarily maximal. Then A_P does not need to be a field.

For P as above, let K_P be the quotient field of A_P.

Definition 11 *Let K be a field properly extending $\mathbb{R}$. K is* super-real *if and only if K is isomorphic (via a map fixing $\mathbb{R}$) to some K_P where P is prime in some $C(X)$.*

So every hyper-real field, and in particular every ultrapower of $\mathbb{R}$, is a super-real field. In fact, we can restrict our attention to the case where the underlying space X is compact:

Theorem 14 *Let $\mathcal{M}$ be a maximal ideal in $C(X)$, and let $K = \mathcal{A}_{\mathcal{M}}$. Let $P = C^b(X) \cap \mathcal{M}$, considered as a z-ideal in $C(\beta X)$. Then $K \cong K_P$.* □

`Unless otherwise stated, from now on` $X = \Omega$ `is assumed to be compact.`

Theorem 15 *Let P be prime in $C(\Omega)$. Then there is an $x_P \in \Omega$ such that*

$$\{x_P\} = \bigcap \mathsf{Z}[P].$$

Proof Let $h(P) = \bigcap \mathsf{Z}[P]$. Then $h(P)$ is non-empty, since $\mathsf{Z}[P]$ has the finite intersection property. Suppose $x_1, x_2 \in h(P)$ are different.

Let U_1, U_2 be neighborhoods of x_1, x_2, respectively, such that $\overline{U}_1 \cap \overline{U}_2 = \emptyset$.

Let $f_1, f_2 \in C(\Omega)$ be such that $f_i(x_i) = 1$ and $f_i|(\Omega \setminus U_i) = 0$. Then $f_1 f_2 = 0$, but $f_1, f_2 \notin P$, contradiction. □

Definition 12 *A totally ordered set* $(\mathbb{P}, \leq)$ *is semi-*η_1 *if and only if it has no* (ω, ω)*-gaps: Whenever* $\mathbb{P} \models s_1 < s_2 < \cdots < t_2 < t_1$*, there is some* $s \in \mathbb{P}$ *such that for all* n, m*,*

$$\mathbb{P} \models s_n < s < t_m.$$

So $\mathbb{R}$ is semi-η_1, but not η_1.

Theorem 16 *Let* P *be prime in* $C(\Omega)$*. Then* $\mathcal{A}_P$ *is semi-*η_1*.*

Proof Given sequences (a_n) and (b_n) in $\mathcal{A}_P$ such that $a_1 < a_2 < \cdots < t_2 < t_1$, inductively choose $f_n, g_n \in C(\Omega)$ such that $f_1 < f_2 < \cdots < g_2 < g_1$, $\pi_P(f_n) = a_n$ and $\pi_P(g_n) = t_n$.

Let $x = x_P$. We may also assume that $f_n(x) \leq 0 \leq g_n(x)$ for all n.

Since the result is clear if $f_n(x) < 0 < g_n(x)$ for all n, suppose (without loss) that $f_m(x) = 0$ for all x.

Set

$$f = f_1 + \sum_k (f_{k+1} - f_k) \wedge 2^{-k}.$$

Then $f \in C(\Omega)$ and $a = \pi_P(f)$ interpolates (a_n) and (b_n). $\square$

Theorem 17 *Let Ω be compact and let P be prime in $C(\Omega)$. Then K_P is real-closed.*

Proof A real field K is real-closed if and only if its complexification $K(i)$ is algebraically closed.

It is easy to see that the correspondence

$$R \mapsto R_{\mathbb{C}} = \{\, f \in C(\Omega, \mathbb{C}) : |f| \in R \,\}$$

is an inclusion-preserving bijection between the set of prime ideals of $C(\Omega)$ and that of $C(\Omega, \mathbb{C})$. It is also easy to see that $K_P(i)$ is isomorphic to the quotient field of $\mathcal{A}_{P_{\mathbb{C}}}(\mathbb{C})$.

Finally, it is not too hard to see that $K_P(i)$ is algebraically closed if and only if every *monic* $p \in \mathcal{A}_{P_{\mathbb{C}}}(\mathbb{C})[X]$ has a root in $\mathcal{A}_{P_{\mathbb{C}}}(\mathbb{C})$.

Let $f_0, \ldots, f_{n-1} \in C(\Omega, \mathbb{C})$, and for $x \in \Omega$ let

$$p_x = f_0(x) + f_1(x)X + \cdots + f_{n-1}(x)X^{n-1} + X^n,$$

so $p_x \in \mathbb{C}[X]$.

Using Rouché's theorem it can be shown that if $z_1(x), \ldots, z_n(x)$ are the roots of p_x, listed so that

$$\Re z_1(x) \leq \cdots \leq \Re z_n(x),$$

then the functions $r_k = \Re z_k$ are continuous on Ω.

A similar argument with the imaginary parts s_k allows us to define continuous functions by

$$u_{j,k}(x) = p_x\big(r_j(x) + s_k(x)\big).$$

By definition, for all $x \in \Omega$ there is some j, k such that $z = r_j(x) + s_k(x)$ is a root of p_x. Thus,

$$\prod_{j,k} u_{j,k} = 0$$

and since $P_{\mathbb{C}}$ is prime, some $u_{j,k} \in P_{\mathbb{C}}$.

Setting $a = \pi_{P_{\mathbb{C}}}(r_j + s_k)$, and $p =$

$\pi_{P_{\mathbb{C}}}(f_0) + \pi_{P_{\mathbb{C}}}(f_1)X + \cdots + \pi_{P_{\mathbb{C}}}(f_{n-1})X^{n-1} + X^n$,

it follows that $p(a) = \pi_{P_{\mathbb{C}}}(u_j, k) = 0$, and we are done. $\square$

However, not all properties of hyper-real fields are shared by all the super-real fields:

Theorem 18 *There is a compact space Ω, and a prime z-ideal P in $C(\Omega)$ such that K_P is not semi-η_1.* □

On the other hand,

Theorem 19 *Let X be discrete space, and let P be a prime ideal in $C(\beta X)$. Then K_P is semi-η_1.*
□

V. Automatic Continuity

All hyper-real fields are real-closed fields and η_1-sets. For a while it was thought that every real-closed η_1-field is a hyper-real field. This is not the case:

Definition 13 *A prime ideal P in $C(\Omega)$ is* exponential *if and only if for every $g \in P^+$ with $g < 1$, it is the case that $1/\ln(1/g) \in P$.*

Notice every maximal ideal is exponential.

Definition 14 *Let K be an ordered field and let G be a convex subgroup of K. An* exponentiation *on G is a map*

$$\exp : G \to K^+ \setminus \{0\}$$

such that for all $a, b \in G$,

1. $\exp(a + b) = \exp(a)\exp(b)$,
2. $\exp(0) = 1, \exp(1) = e$,
3. *$a < b$ implies $\exp(a) < \exp(b)$,*
4. *the range of* exp *is $K^+ \setminus \{0\}$.*

Theorem 20 *Let P be prime in $C(\Omega)$. Then there is an exponential on a convex subgroup G of K_P. If P is exponential, then we can take $G = P$.*

□

The proof of the theorem involves the development of an *operational calculus* for super-real fields.

Corollary 2 *Let K be hyper-real. Then there is an exponentiation on K.* □

Theorem 21 *There are real-closed η_1-fields with no exponentiations in any of their convex subgroups. Thus, these fields are not super-real.*

□

Theorem 22 *(Kaplansky, 1949) If $\|\cdot\|$ is an arbitrary algebra norm on $C(\Omega, \mathbb{C})$, then for any $f \in C(\Omega, \mathbb{C})$,*

$$\|f\|_\Omega \leq \|f\|. \quad \square$$

So, if $\|\cdot\|$ is an algebra norm on $C(\Omega, \mathbb{C})$, then $\|\cdot\|$ is equivalent to $\|\cdot\|_\Omega$ if and only if for all $f \in C(\Omega, \mathbb{C})$ there is an $M > 0$ such that

$$\|f\| \leq M\|f\|_\Omega.$$

This happens, for example, whenever $\|\cdot\|$ is complete (by the open mapping theorem).

Kaplansky's problem: Is every algebra norm on $C(\Omega, \mathbb{C})$ complete?

Theorem 23 *Every algebra norm on $C(\Omega, \mathbb{C})$ is complete if and only if every homomorphism from $C(\Omega, \mathbb{C})$ into a Banach algebra is continuous.*

Proof If $\|\cdot\|$ is not complete, and $\mathcal{A}$ is the completion of $(C(\Omega, \mathbb{C}), \|\cdot\|)$, then $\mathcal{A}$ is a unital commutative Banach algebra, and the inclusion

$$i : C(\Omega, \mathbb{C}) \to \mathcal{A}$$

is discontinuous.

Conversely, if $\mathcal{B}$ is a Banach algebra and

$$\theta : C(\Omega, \mathbb{C}) \to \mathcal{B}$$

is a discontinuous homomorphism, then

$$f \mapsto \|f\| = \max\{\|f\|_\Omega, \|\theta(f)\|\}$$

is an incomplete norm on $C(\Omega, \mathbb{C})$. $\square$

Theorem 24 *There is a discontinuous homomorphism from $C(\Omega, \mathbb{C})$ into a Banach algebra if and only if there is a prime ideal P in $C(\Omega)$ such that $\mathcal{A}_P$ is normable.* □

Theorem 25 *(Dales; Esterle) Assume* CH. *Let P be a nonmaximal prime ideal in $C(\Omega, \mathbb{C})$ such that $|\mathcal{A}_P| = \mathfrak{c}$. Then $\mathcal{A}_P$ is normable.* □

Under which conditions is $\mathcal{A}_P$ normable?

Definition 15 *Let K be an ordered field extending $\mathbb{R}$.*

$$K^{fin} = \{ a : (\exists n \in \mathbb{N})\, |a| \leq n \}.$$

Notice that $\mathcal{A}_P \subset K_P^{fin}$, and that K_P^{fin} is an algebra.

Open question:

- For which compact spaces Ω and prime ideals P in $C(\Omega)$ is K_P^{fin} normable?

Definition 16 *Let K be an ordered field. Its* value set *is*

$$\Gamma_K = (K \setminus \{0\}) / \sim,$$

where $a \sim b$ if and only if for some $n, m \in \mathbb{N}$,

$$|a| \leq n|b| \leq m|a|.$$

Theorem 26 *(Esterle) If K is an ordered field, and K^{fin} is normable, then $|\Gamma_K| \leq \mathfrak{c}$, and $|K| \leq 2^{\mathfrak{c}}$.* $\square$

Theorem 27 *(CH) Let* $\mathcal{U}$ *be a nonprincipal ultrafilter on* $\mathbb{N}$. *Then*

$$(\mathbb{R}^{\mathbb{N}}/\mathcal{U})^{fin}$$

is normable, so there are discontinuous homomorphisms from $C(\beta\mathbb{N}, \mathbb{C})$ *into a Banach algebra.* □

Theorem 28 *(Woodin) It is consistent with* MA *that every homomorphism from any* $C(\Omega, \mathbb{C})$ *into any Banach algebra is continuous.* □

Theorem 29 *(Todorcevic) Assume* PFA. *Then every homomorphism from any* $C(\Omega, \mathbb{C})$ *into any Banach algebra is continuous.* □

Theorem 30 *Assume* GCH. *Let K be an ordered field.*

1. *If $|K| > \aleph_2$ or $|\Gamma_K| > \aleph_1$ then K^{fin} is not normable.*
2. *If $|K| = \aleph_1$, then K^{fin} is normable.*
3. *If $|K| = \aleph_2$ then $|\Gamma_K| \geq \aleph_1$.* □

Theorem 31 *Assume* GCH. *There is a compact space Ω and a non-maximal, prime z-ideal P in $C(\Omega)$ such that K_P is an η_1-field, $|K_P| = \aleph_2$ and $|\Gamma_K| = \aleph_1$.* □

Theorem 32 *It is consistent, relative to the existence of almost huge cardinals, that there is an ultrafilter $\mathcal{U}$ on ω_1 such that if $K = \mathbb{R}^{\omega_1}/\mathcal{U}$, then*

$$|K| = \aleph_2 \quad \text{and} \quad |\Gamma_K| = \aleph_1. \quad \Box$$

We close with an open problem:

Let K be any ordered field with $|K| = \aleph_2$ and $|\Gamma_K| = \aleph_1$. Determine whether K^{fin} is normable.

References

1. M.Y. Antonovskij, D.V. Chudnovsky, G.V. Chudnovsky, E. Hewitt, *Rings of real-valued continuous functions. II*, Math. Zeit., 151–186, 1981.

2. H.G. Dales, W.H. Woodin, **An introduction to independence for analysts**, LMS lecture note series **115**, 1987.

3. H.G. Dales, W.H. Woodin, **Super-real fields**, LMS monographs **14**, 1996.

4. L. Gillman, M. Jerison, **Rings of continuous functions**, GTM **43**, reprint of the 1960 ed.

5. W. Rudin, **Functional Analysis**, McGraw-Hill, 1991.

6. S. Shelah, R. Jin, *Possible size of an ultrapower of* ω, Arch. Math. Log., accepted.