Stationary subsets of ω_1 and models of set theory

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Boise State University, February 22nd, 2005

§0. Abstract

We survey some classical and new results related to the *preservation* of stationary subsets of ω_1 between models of set theory.

The new results we mention are part of joint work with Paul Larson and Boban Veličković.

§1. First order logic

In *(first order) logic* we start by specifying a *language* $\mathscr{L}$, and proceed to study the properties of $\mathscr{L}$*-structures.*

The language $\mathscr{L}$ consists of a collection of constant symbols

$$c_0, c_1, \ldots,$$

relation symbols

$$R_0, R_1, \ldots,$$

each of a specific (finite) arity, and function symbols

$$f_0, f_1, \ldots,$$

also of a specific (finite) arity.

§1. First order logic

An $\mathscr{L}$-structure $\mathscr{M}$ consists of a set M and a collection of elements of M, relations on M, and functions on M, corresponding to the symbols of the language.

For example, if $R \in \mathscr{L}$ is a binary relation symbol, then the corresponding relation on M is $R^{\mathscr{M}} \subseteq M \times M$, and if $f \in \mathscr{L}$ is a ternary function symbol, then $f^{\mathscr{M}} : M^3 \to M$.

Although this can be done in abstract, we usually have in mind a specific collection of structures we expect to study, and the choice of the language is intended to reflect some of the features the study will focus on.

For example, to study groups, we may start with the language

$$\mathscr{L}_1 = \{f\},$$

where f is a symbol for a binary function (intended to represent the group operation), but we may as well start with the language

$$\mathscr{L}_2 = \{c, f\}$$

where, in addition, we have a constant symbol in the language (intended to represent the identity of the group).

§1. First order logic

Once the language $\mathscr{L}$ is specified, we consider the *formulas* of this language. These are the statements that can be formed using *variables*

$$x, y, z, \ldots,$$

the symbols of $\mathscr{L}$, the logical connectives

$$\wedge, \vee, \neg, \rightarrow, \leftrightarrow,$$

and the quantifiers

$$\exists, \forall.$$

Given a formula φ and an $\mathscr{L}$-structure $\mathscr{M}$, we can formalize the concept "*φ holds in $\mathscr{M}$*", or "*φ is true in* (or, *of*) *$\mathscr{M}$*". We will skip the formal definition, but the meaning is the obvious one.

§1. First order logic

For example, consider a language with a binary relation symbol R and a constant symbol c. In this language we can form the statement

$$\forall x\,(xRc).$$

This formula holds in $\mathbb{N}$ if we interpret c as 1 and R as "is divisible by". It is false in $\mathbb{N}$ if we interpret c as 0 and R as $>$, etc.

The meaning of a statement, and whether it holds or not, may very well change as we change the structure under consideration.

§1. First order logic

A *(first order) theory* is specified by giving a collection of formulas (the axioms of the theory) in some language $\mathscr{L}$.

The *models* of the theory are the $\mathscr{L}$-structures in which the axioms hold. A deep theorem of Gödel (*completeness*) states that we can prove a statement from the axioms iff the statement holds in all the models of the theory.

Group theory and field theory are examples of first order theories. *First order* refers to the fact that we only allow quantification over elements of the structure, but not over its subsets. Thus, even though most of group theory deals with the relations between groups G and their subgroups, the formal language itself does not allow us to express statements about subgroups of G.

§2. Set Theory

We work in the standard system of set theory (usually referred to as ZFC, Zermelo-Fraenkel with the Axiom of Choice). This is a first order theory in the language $\mathscr{L} = \{\in\}$ whose only symbol is one for a binary relation, although the specific details of this theory are not required for what follows.

It suffices to know that the axioms of ZFC describe *rules of set formation* from which one can deduce that

1. There are (infinite) sets.
2. The collection of all sets is closed under basic operations, like taking unions, power sets, etc.

These rules of set formation are powerful enough that for any mathematical structure (to date), e.g., groups, Banach algebras, manifolds, etc, there is a *surrogate set* (i.e., an isomorphic copy) built up from pure sets by means of these rules. (This is a small white lie: *Categories* can be represented by means of pure sets but are not sets themselves.)

This is (mostly) what is meant by the claim that *set theory provides an adequate foundation for all of mathematics.*

Set theorists investigate the properties of sets (pure or coming from other mathematical disciplines) that these rules determine. Part of this investigation consists of studying those properties that the rules are not strong enough to decide.

§2. Set Theory

For example, none of the following questions are decided by the axioms of set theory:

1. The *Continuum Hypothesis* CH.
2. Questions of automatic continuity in Banach algebra theory.
3. Questions about the Lebesgue measurability of certain sets of reals.

Let us mention that one of the axioms of ZFC, *regularity*, is equivalent to the claim that $\in$ is *well-founded*, i.e., there is no infinite descending $\in$-chain

$$x_0 \ni x_1 \ni x_2 \ni \dots$$

§3. Ordinals

Definition 3.1. X is *transitive* iff any element of X is a subset of X:

$$x \in y \in X \Rightarrow x \in X.$$

Definition 3.2. α is a (von Neumann) *ordinal* iff

1. α is transitive, and
2. Every element of α is transitive.

If α is an ordinal, then $\in$ is a linear order of α. By regularity, $(\alpha, \in)$ is *well-ordered*, i.e., every nonempty subset of α has an $\in$-first element.

The *Axiom of Choice* is equivalent to the statement

> For every X there is an ordinal α and a bijection $f : X \to \alpha$.

§3. Ordinals

Notice that the elements of ordinals are ordinals themselves. Any two ordinals are $\in$-comparable (i.e., if α and β are distinct ordinals, then either $\alpha \in \beta$ or $\beta \in \alpha$). Thus, the *class* ORD of all ordinals is well-ordered by $\in$.

Given X, the smallest α such that there is a bijection $f : X \to \alpha$ is called the *cardinality* of X and denoted $|X|$.

For example:

- $\emptyset$ is an ordinal.
- If α is an ordinal, $\alpha + 1 := \alpha \cup \{\alpha\}$ is an ordinal. It is the smallest ordinal β such that $\alpha \in \beta$.

§3. Ordinals

We can identify the ***natural numbers*** with the finite ordinals, thus

- $0 = \emptyset$,
- $1 = \{\emptyset\}$,
- $2 = \{\emptyset, \{\emptyset\}\}$,
- In general, $n + 1 = \{0, 1, \ldots, n\}$.

We denote by ω the set of finite ordinals; ω is an ordinal, and is the set-theoretic counterpart of the natural numbers $\mathbb{N}$,

$$\omega = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \ldots\} = |\mathbb{N}|.$$

§4. ω_1

There are many countable ordinals:

$$0, 1, 2, \ldots, \omega, \omega + 1, \omega + 2, \ldots, \omega + \omega = \omega 2,$$
$$\omega 2 + 1, \ldots, \omega 3, \ldots, \omega^2, \ldots$$

Cantor showed that for every X, $|\mathscr{P}(X)| > |X|$, so there is a (first) uncountable ordinal, we denote it by ω_1.

§4. ω_1

Another description of ω_1:

Consider the collection Y of all well-orders $(\mathbb{N}, <)$ of the natural numbers. Define an equivalence relation on Y by

$$<_1 \cong <_2 \text{ iff}$$

there is a permutation $\pi : \mathbb{N} \to \mathbb{N}$ such that

$$\pi(<_1) = <_2 .$$

§4. ω_1

Let $X = Y/\cong$. Define a relation $\prec$ on X by:

$$[<_1] \prec [<_2] \text{ iff}$$

there is an injection $\rho : \mathbb{N} \to \mathbb{N}$ sending $<_1$ to a proper initial segment of $<_2$.

This is well-defined (i.e., it does not depend on the representatives of the equivalence classes) and is a linear ordering of X.

In fact, $(X, \prec)$ is well-ordered. X is uncountable, and for every $a \in X$,

$$\{ b \in X : b \prec a \}$$

is countable.

Then $\omega_1 = |X|$ and, in fact, $(X, \prec)$ is order-isomorphic to $(\omega_1, \in)$.

§5. Club sets

There are three kinds of ordinals:

- $\emptyset$.
- *Successor ordinals*: Those of the form $\alpha + 1$ for some α.
- *Limit ordinals*: Those that are not successors.

For example, $\omega, \omega + \omega, \omega_1$ are limit ordinals.

For ordinals α, β, we write $\alpha < \beta$ iff $\alpha \in \beta$, so an ordinal is the set of its predecessors:

$$\alpha = \{ \beta : \beta < \alpha \}.$$

§5. Club sets

Then $(\omega_1, <)$ is a linearly ordered set, and we consider it a topological space with the order topology:

Basic open sets have the form

- $\alpha = \{\, \beta : \beta < \alpha \,\}$ $(\alpha \leq \omega_1)$, or
- $\omega_1 \setminus (\alpha + 1) = \{\, \beta < \omega_1 : \beta > \alpha \,\}$.

Notice that if α is not a limit ordinal then $\{\alpha\}$ is open.

If α is limit, $\mathscr{O}$ is open, and $\alpha \in \mathscr{O}$, then $\mathscr{O}$ contains an interval $(\beta, \alpha]$ for some $\beta < \alpha$.

(It follows, for example, that $\alpha = \{\, \beta : \beta < \alpha \,\}$ is *compact* iff α is not limit.)

§5. Club sets

We now introduce a key notion that will correspond to the intuitive idea of a *large subset of* ω_1. A good analogy is the concept of a *full measure subset of* $[0, 1]$.

Definition 5.3. $C \subseteq \omega_1$ is *club* iff

1. C is closed in the order topology (i.e., if $\alpha > 0$ is limit and $C \cap \alpha$ is unbounded in α, then $\alpha \in C$).
2. C is unbounded (in the order of ω_1).

Notice that $X \subseteq \omega_1$ is *unbounded* iff X is uncountable, because (by the Axiom of Choice) a countable union of countable sets is countable.

§5. Club sets

We have the following facts:

- The intersection of two clubs is club.
- In fact, the intersection of countably many clubs is club.
- If $A \subseteq \omega_1$, let $A' = \{\beta \in \omega_1 : \beta$ is a limit point of elements of $A\}$. If C is club, then $C' \subseteq C$ is club.

For example, the following sets are clubs:

- ω_1.
- $\omega_1' = \{\alpha < \omega_1 : \alpha$ is a limit ordinal$\}$.
- $\omega_1'' = \{\alpha < \omega_1 : \alpha$ is a limit of limit ordinals$\}$,

etc.

§5. Club sets

- If C is a club, $C = \{\alpha_0 < \alpha_1 < \alpha_2 < \dots\}$ then
$$\{\beta < \omega_1 : \alpha_\beta = \beta\}$$
is club.

- If $f : \omega_1 \to \omega_1$, then
$$\{\beta < \omega_1 : \forall\alpha < \beta\,(f(\alpha) < \beta)\}$$
is club.

§6. Stationary sets

Together with the intuitive notion of big sets, we have the intuitive notion of medium sized sets (in the measure theoretic analogue, these correspond to sets of positive measure).

Definition 6.4. $S \subseteq \omega_1$ is *stationary* iff $S \cap C \neq \emptyset$ for any club C.

Any stationary set S is unbounded, and in fact if S is stationary and C is club, then $S \cap C$ is stationary.

The first significant use of the Axiom of Choice in the study of ω_1 comes in the following result:

Fact 6.5 (Ulam). *There are disjoint stationary subsets of* ω_1.

§6. Stationary sets

That the use of Choice is essential here can be argued as follows:

From natural extensions of ZFC one can show that if M is the collection of all sets *constructible* using reals and ordinals (M is usually called $L(\mathbb{R})$), then

- M contains all sets relevant to analysis.
- All subsets of the reals in M are Lebesgue measurable.
- The structure $(M, \in)$ satisfies all the axioms of ZFC except the Axiom of Choice, and also satisfies a weak version of Choice, DC, sufficient for all of classical mathematics.
- Given any $X \subseteq \omega_1$, if $X \in M$ then either X or $\omega_1 \setminus X$ contains a club.

§6. Stationary sets

In fact, from Choice one can prove

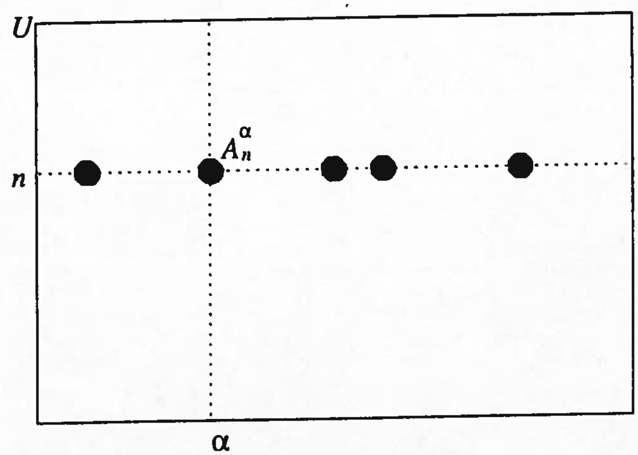
Fact 6.6 (Ulam). *Any stationary sets S can be partitioned into ω_1 many disjoint stationary subsets.*

Proof: Any $\alpha < \omega_1$ is countable, so we can fix for each $\alpha \in S$ an injection $\iota_\alpha : \alpha \to \mathbb{N}$.

For each $\alpha < \omega_1$ and $n \in \mathbb{N}$ let

$$A_n^\alpha = \{\beta \in S : \alpha < \beta \text{ and } \iota_\beta(\alpha) = n\}.$$

§6. Stationary sets



An Ulam matrix.

For each n, notice that if $\alpha \neq \gamma$ then $A_n^\alpha \cap A_n^\gamma = \emptyset$.

Also, $\bigcup_n A_n^\alpha = \{ \beta \in S : \beta > \alpha \} = S \cap [\alpha + 1, \omega_1)$ is stationary.

Then, *for any* α, there is n_α such that $A_{n_\alpha}^\alpha$ is stationary.

(**Proof:** If T_n is not stationary, $n \in \mathbb{N}$, and $T = \bigcup_n T_n$, then T is not stationary: Let C_n be club, $C_n \cap T_n = \emptyset$. Let $C = \bigcap_n C_n$. Then $C \cap T = \emptyset$. □)

There must be some n such that for an uncountable set A of α, $n_\alpha = n$.

Then for all $\alpha \in A$, all the A_n^α are stationary, they are all disjoint, and they are all subsets of S. $\square$

§7. Models of set theory

We will use the phrase *model of set theory* or, simply, *model* to refer to a specific class of models M of ZFC. These models must, in addition, satisfy the following three requirements:

- The interpretation $\in^M$ of the membership relation is membership, i.e.,
$$\in^M = \in \cap (M \times M).$$
- M is transitive.
- $\mathrm{ORD} \subset M$.

For example, in 1940 Gödel introduced the class L of *constructible* sets. There is a formula φ such that

$$L = \{ x : \varphi(x) \}$$

is a model of set theory, and such that given *any* model M, $L \subseteq M$.

§7. Models of set theory

In 1963 Cohen (and then Solovay) showed that given any model M and any partial order $\mathbb{P} \in M$ (called a *forcing notion*) we can form a new model $M^{\mathbb{P}}$ such that $M \subseteq M^{\mathbb{P}}$ and the (order theoretic) properties of $\mathbb{P}$ determine which statements hold in $M^{\mathbb{P}}$.

This method (called *forcing*) is incredibly flexible, and has allowed set theorists to show that many statements (like CH) are neither provable nor refutable from ZFC.

[I am being slightly imprecise here: If M is the collection V of *all* sets, obviously there is no bigger model $V^{\mathbb{P}}$. We can still make sense of forcing in this case, but the formalization is quite technical.]

§7. Models of set theory

In what follows, we are mostly interested in pairs of models $M \subseteq V$ such that the ordinal ω_1 is the same from the point of view of M as from the point of view of V.

The point is: An ordinal α is uncountable iff there is no surjection $f : \omega \to \alpha$. So for an ordinal α to be ω_1 from the point of view of M (we write $\alpha = \omega_1^M$) it is necessary that there is no such surjection f *in* M. Of course, such an f may very well exist in V, since V contains more sets. For example, from natural extensions of ZFC one can show that ω_1^L is countable.

§7. Models of set theory

So, suppose $M \subseteq V$ are models and $\omega_1^M = \omega_1^V$.

- If $C \subseteq \omega_1^M$ is club in M, then it is club in V.
- On the other hand, if S is stationary in M, S may or may not be stationary in V. In fact, given *any* S stationary in M there is a partial order $\mathbb{P}_S \in M$ such that
  - $\omega_1^M = \omega_1^{M^{\mathbb{P}}}$ and in fact, given any $f : \omega \to \mathrm{ORD}$, if $f \in M^{\mathbb{P}}$, then $f \in M$.
  - S contains a club in $M^{\mathbb{P}_S}$.

Let T be the complement of S, $T = \omega_1^M \setminus S$. We have seen (from Ulam's theorem) that T may also be stationary. But clearly $T \cap S = \emptyset$, so T is not stationary in $M^{\mathbb{P}_S}$.

- Nevertheless, if $S \in M$, S is stationary in V, $S = A \cup B$, $A, B \in M$, then at least one of A, B is stationary in V. Similarly for countable unions.

Can it be the case that *for every* $S \subseteq \omega_1^M$, $S \in M$, either S or $\omega_1^M \setminus S$ contains a club in V? [This happens when M is the 'choiceless' model $L(\mathbb{R})$ that we mentioned earlier.]

NO. Notice that the Ulam matrix we discussed earlier can be built in M (so all the sets A_n^α are in M) but it can be analyzed in V. So:

> Given any $S \in M$, S stationary in V, there are ω_1 many disjoint subsets $A_\alpha \subset S$, $\alpha < \omega_1$, such that each $A_\alpha \in M$ and A_α is stationary in V.

§7. Models of set theory

But, in general, the *enumeration*

$$\mathscr{A} = \langle A_\alpha : \alpha < \omega_1 \rangle$$

may not be in M. In fact,

- It can be the case that no infinite subsequence of $\mathscr{A}$ is in M.
- It is not even clear that there is a *partition* of S into ω_1 many pieces that belong to M and are stationary in V, even if the enumeration itself is not in M.

Notice that if a countable subset B of ω_1^M belongs to V but not to M, then there are reals in V that are not in M. Denote by $\mathbb{R}^M$ and $\mathbb{R}^V$ the set of reals from the point of view of M and of V, respectively.

So suppose now that $\mathbb{R}^M = \mathbb{R}^V$. Ulam's result tells us that for any $S \in M$ that is stationary in V, there is (in M) a partition of S into countably many disjoint subsets such that all of them are stationary in V. Can we find ω_1 many?

§8. Ramsey theory

Given $f : \mathbb{N} \to n$, $n \in \mathbb{N}$

$$(\text{i.e., } f : \mathbb{N} \to \{0, 1, \ldots, n-1\}),$$

there is some $i < n$ such that $f^{-1}(\{i\})$ is infinite.

Infinite Ramsey theory is concerned with generalizations of this observation.

Definition 8.7. Given $A \subseteq \omega_1$, let

$$[A]^2 = \{\, (\alpha, \beta) \in A \times A : \alpha < \beta \,\}.$$

Theorem 8.8 (Ramsey). *Given any $n \in \mathbb{N}$ and any $f : [\mathbb{N}]^2 \to n$, there is $i < n$ and an infinite $H \subseteq \mathbb{N}$ such that*

$$\forall (a, b) \in [H]^2 \, (f(a, b) = i).$$

§8. Ramsey theory

A natural question to ask is whether the same holds of ω_1, i.e., given $f : [\omega_1]^2 \to 2$, is there $H \subseteq \omega_1$ uncountable, and an $i < 2$ such that

$$\forall(\alpha, \beta) \in [H]^2\,(f(\alpha, \beta) = i)?$$

In 1965, Paul Erdős, András Hajnal and Eric Milner gave a strong form of a *negative* answer, assuming CH.

In 1985, Stevo Todorčević gave a weak form of their negative answer without any additional assumptions.

Last year, Justin Moore gave a stronger form of Todorčević's negative answer.

§8. Ramsey theory

Using the CH argument, we can show:

Fact 8.9. *If* CH *holds in* $M \subseteq V$ *and* $\dot{\mathbb{R}}^M = \mathbb{R}^V$, *then any* $S \in M$ *that is stationary in* V *can be partitioned in* M *into* ω_1 *many disjoint subsets all of which are stationary in* V.

Using Todorčević's original argument we can show:

Fact 8.10. *If* $M \subseteq V$, $\omega_1^M = \omega_1^V$, *then any* $S \in M$ *that is stationary in* V *can be partitioned in* M *into countably many disjoint subsets all of which are stationary in* V.

On the other hand, it is not always possible to find such a partition into ω_1 many pieces:

Fact 8.11 (Larson). *Given any model M, there is a $\mathbb{P} \in M$ such that $\omega_1^M = \omega_1^{M^{\mathbb{P}}}$ but for any sequence $\langle S_\alpha : \alpha < \omega_1^M \rangle \in M$ of disjoint sets, at least one of them is not stationary in $M^{\mathbb{P}}$.*

It is still open whether such a partition must exist whenever $\mathbb{R}^M = \mathbb{R}^V$.

§9. BPFA

Let's now consider a different situation, where $M \subseteq V$, $\omega_1^M = \omega_1^V$ and *all* stationary sets in M are stationary in V.

Definition 9.12. Given a set X, the *transitive closure of* X, $\mathrm{tc}(X)$, is the smallest transitive set Y such that $X \in Y$.

For A a set, let $\bigcup A = \bigcup\{\, a : a \in A \,\}$. One can show that

$$\mathrm{tc}(X) = \{X\} \cup X \cup \bigcup X \cup \bigcup\bigcup X \cup \ldots .$$

Definition 9.13. $H_{\omega_2} = \{\, X : |\,\mathrm{tc}(X)| \leq \omega_1 \,\}$.

§9. BPFA

A statement $\varphi(x, y)$ is *bounded* iff it is *first order* in the structure

$$(\mathrm{tc}(\{x, y\}), \in).$$

Bounded statements φ have the property that their meaning is ***absolute***, so if $x, y \in M \subseteq V$, then $\varphi(x, y)$ holds in M iff $\varphi(x, y)$ holds in V.

For example, the following are bounded statements:

- C is closed and unbounded in α.
- f is a function, $\mathrm{dom}(f) = \omega$, $\mathrm{ran}(f) \subseteq \omega + 1$.
- α is a limit ordinal.

§9. BPFA

We will not define *properness.* It is a technical condition on a partial order $\mathbb{P} \in M$ that guarantees that all stationary sets in M are stationary in $M^{\mathbb{P}}$.

Definition 9.14 (Goldstern, Shelah).
The *Bounded Proper Forcing Axiom* BPFA holds in M iff

for any $x \in H_{\omega_2}$ and any bounded statement $\varphi(\cdot,\cdot)$,

if there is a proper partial order $\mathbb{P} \in M$ such that in $M^{\mathbb{P}}$,

there is a $y \in H_{\omega_2}$ such that $\varphi(x, y)$ holds,

then there is such a y already in M.

§9. BPFA

Intuitively, one can think of BPFA as a statement about the *richness* of the model M, in the sense that many (existential) statements that are *consistent* (and hold of extensions $M^{\mathbb{P}}$ of M where $\mathbb{P}$ is proper) are actually *true* in M, i.e., M already contains witnesses to all of these statements.

BPFA is not a theorem of set theory, so there are models where it is false. Under natural assumptions, there are models where it is true. BPFA has many applications in combinatorics, set theoretic topology, group theory and analysis. In essence, BPFA is a statement about subsets of ω_1.

How much does this additional assumption determine about the models that satisfy it?

§9. BPFA

Definition 9.15. ω_2 is the first ordinal such that $|\omega_2| > \omega_1$. Given a model M, we write ω_2^M for the ordinal that is ω_2 from the point of view of M.

The standard method to build a model of BPFA starts with a model M and a carefully chosen $\mathbb{P} \in M$ such that BPFA holds in $M^{\mathbb{P}}$. Moreover, $\omega_1^M = \omega_1^{M^{\mathbb{P}}}$, but $\omega_2^M < \omega_2^{M^{\mathbb{P}}}$ (and much more).

Theorem 9.16 (Veličković, C.). *Assume $M \subseteq V$ are models of set theory such that $\omega_1^M = \omega_1^V$ and* BPFA *holds in both M and V. Then either $\omega_2^M < \omega_2^V$ or else* **every** *subset of ω_1^V in V belongs to M.*

The proof of this result uses game theoretic arguments, and methods recently introduced by Justin Moore.

§9. BPFA

The proof allows us to define a bijection $\rho : \mathbb{R} \to \omega_2$, so

$$\mathsf{BPFA} \Rightarrow |\mathbb{R}| = \omega_2,$$

a result previously shown by Moore.

If $M \subseteq V$, BPFA holds in M and V but $\omega_2^V > \omega_2^M$, then in fact $\omega_3^M < \omega_2^V$. Here, ω_3 is the first ordinal such that $|\omega_3| > \omega_2$. It is open precisely how big ω_2^V must be in this case.

§9. BPFA

Let me close by mentioning that even though our argument is different, Moore's original method to prove $|\mathbb{R}| = \omega_2$ from BPFA codes reals using (among other tools) a sequence of disjoint stationary sets.

This was the original motivation for the results on preservation of sequences of stationary sets we mentioned earlier.

§10. References

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3. J. Moore. *Set mapping reflection*, preprint.