Caicedo — 5 Nov — Set Theory Seminar

Aurichi: Selective CCC

"countable steps"

measure zero vs. strong measure zero

α_i

$\sum \alpha_n < \varepsilon$

$\alpha_1 < \varepsilon_1$

$\alpha_2 < \varepsilon_2$

$\vdots$

$\sum \varepsilon_i < \infty$

and α's still cover

Def: Menger property

$\forall (\mathcal{O}_n : n<\omega)\ \exists (\mathcal{F}_n : n<\omega)\ (\bigcup \{\mathcal{F}_n : n<\omega\}$ is a cov

cover of X by open sets

finite subsets of $\mathcal{O}_n$

$S_{fin}(\mathcal{O},\mathcal{O})$

Rothberger

$\forall (\mathcal{O}_n : n<\omega)\ \exists (A_n : n<\omega)\ \{A_n : n\in\omega\}$ is a cover

$\mathcal{O}_n$

$S_1(\mathcal{O},\mathcal{O})$

Heurewicz

Hurewicz compactness

$\forall (\mathcal{O}_n : n<\omega)\ \exists (\mathcal{F}_n : n<\omega)\ \forall x\ x \in \bigcup \mathcal{F}_n\ \forall^{*} n$

all but finitely many

$\mathcal{O}_n$ finite

Separable ⇝ ctbl dense subset

M separable | replace open cover w/ dense subset

$S_{fin}(\mathcal{D},\mathcal{D})$

collection of dense subset

M-separable: $\forall (D_n : n<\omega)\ \exists (F_n : n<\omega) \left(\bigcup \{\quad\} \text{ is dense}\right)$

dense in X

R-separable $\forall (D_n : n<\omega) \exists (p_n : n<\omega) \left(\{p_n : n\in\omega\} \text{ is dense}\right)$

$\in D_n$

H-separable $\forall (D_n : n<\omega) \exists (F_n : n<\omega)\ \forall O$ open

$O \cap F_n \neq \emptyset\ \forall^{\infty} n$

$\subseteq D_n$ finite

2nd ctble (has a ctble basis)

$\Downarrow$ $\Downarrow$

R-sep H-sep

$\Downarrow$ $\Downarrow$

M-sep

$\Downarrow$

sep

eral

o arrows

es

se)

Def:

$\boxed{\text{CCC}}$ A space is CCC if any collection of pairwise disjoint open sets is countable.

Sep $\Rightarrow$ CCC obviously CCC $\not\Rightarrow$ sep.

$\boxed{\text{Selective CCC : R-CCC}}$

$\forall(\mathcal{C}_n : n<\omega)\ \exists(O_n\ n<\omega)\underbrace{(\{O_n : n\in\omega\}\text{ is dense})}_{\bigcup_n O_n \subset X \text{ dense in } X}$ maximal?

$\mathcal{C}_n$: collection of open pairwise disjoint sets (maximal) dense?

$O_n \in \mathcal{C}_n$

Fact: R-CCC $\Rightarrow$ CCC

Pf: Given $\mathcal{A}$ a maximal collection of open pairwise disjoint sets (maximal antichain).

Let $\mathcal{C}_n = \mathcal{A}$ for all n.

Then the $\{O_n : n<\omega\}$ we obtain is all of $\mathcal{A}$

So $\mathcal{A}$ was ctbl to begin with $\boxtimes$

In part. ccc $\not\Rightarrow$ R-ccc (Aurichi's Work)

On the other hand, ccc $\not\Rightarrow$ R-ccc

In fact,

<u>Theorem</u>: $\exists$ zero-dimensional Hausdorff <u>countable</u> space that is not R-ccc

<u>Pf</u>: $X = \omega^{<\omega}$ (finite seq. of $\mathbb{N}$)

We define a top on X.

Let $V_s = \{t \in \omega^{<\omega} : s \subset t\}$ for $s \in \omega^{<\omega}$

basic open nbhd is a $V_s \setminus \bigcup_{t \in F} V_t$ for $s \in \omega^{<\omega}$ and $F \subset V_s$ is s.t.

$\forall n.$ $\{t \in F : \operatorname{dom} t = n\}$ is finite.

① X is Hausdorff

② X is zero-dim

③ Not Sel-ccc:

Let $\mathcal{C}_n = \{V_t : t \in \omega^{<\omega} \operatorname{dom} t = n\}$

* if $t \neq s$ $\operatorname{dom} t = \operatorname{dom} s$ $V_t \cap V_s = \emptyset$

* $\bigcup \mathcal{C}_n$ is dense (easy)

Given $V_s \setminus \bigcup_{t \in F} V_t$ Let $m = \operatorname{dom} s$

If $m < n$. Pick $q \in \omega^{<\omega}$ and $\operatorname{dom} q = n$

$q \notin F$ $\emptyset \neq V_q \cap (V_s \setminus \bigcup_{t \in F} V_t)$

If $m \geq n$ Then

if $q = s \restriction n$ then $V_q \cap (V_s \setminus \bigcup_{t \in F} V_t) \neq \emptyset$

So it's dense.

We pick an element from each

If $V_{p_n} \in \mathcal{C}_n$ then $\bigcup_n V_{p_n}$ is not dense.

Let $p \in \omega' \neq p_1$ then $V_p \setminus \bigcup_n V_{p_n} \neq \emptyset$ $\boxtimes$

$V \cap W \cap Y = \emptyset \Rightarrow V \cap W = \emptyset$

$\overset{?}{\Leftarrow}$

① Fact: $Y \subset X$, Y dense. X is R-ccc iff Y is R-ccc

② continuous open images of R-ccc spaces are R-ccc.

Suppose $f: X \to Y$ is cont. & onto & open

Given $(\mathcal{A}_n : n<\omega)$. Let $\mathcal{B}_n = \{f^{-1}[A] : A \in \mathcal{A}_n\}$

So $(\mathcal{B}_n : n<\omega)$ is a collection max-antichains with $\bigcup_n \mathcal{B}_n$ dense.

then $\{A_n = f[B_n] : n<\omega\}$ works

③ R-sep $\Rightarrow$ R-ccc

$(\mathcal{A}_n : n<\omega)$ For each n $\bigcup \mathcal{A}_n = D_n$

R sep gives $(p_n : n \in \omega)$ $p_n \in D_n$

Pick $A_n \in \mathcal{A}_n$ with $p_n \in A_n$

and $\bigcup_n A_n$ is dense. ☒

Cor: $\mathbb{R}$ is R-ccc because it is R-sep.

Pf. Let $\mathbb{Q} \subseteq \mathbb{R}$ be dense, ctbl $= \{q_n : n<\omega\}$ where $\forall q \in \mathbb{Q}$ $\exists^\infty_n (q = q_n)$

Given $(D_n : n<\omega)$ each D_n dense

Pick $p_n \in D_n$ who $|p_n - q_n| < \frac{1}{n}$.

Then $\{p_n : n \in \omega\}$ is dense. ☒

But now we want stuff thats R-ccc but not inherently

Prop: If κ is a cardinal, 2^κ is R-ccc.

Note that 2^κ, $\kappa > \mathfrak{c}$ ← continuum. is not separable

So this would be an example of R-ccc but not R-sep

Pf: We use an analogue of the classical

$\boxed{\text{Thm}}$: If X_ξ is ccc $\forall \xi < \kappa$ and $\prod_{\xi \in F} X_\xi$ is ccc $\forall F$ finite then $\prod_{\xi < \kappa} X_\xi$ is ccc.

Namely: Fact: If X_ξ is R-ccc $\forall \xi$ and $\prod_{\xi \in S} X_\xi$ is R-ccc for all S ctble, then $\prod_\xi X_\xi$ is R-ccc

(Note converse is true) Same proof as before.

Pf: Note in particular, that $\prod_\xi X_\xi$ is ccc Let $(\mathcal{A}_n : n < \omega)$ be a collection of max antichains in the product.

Wma: each $\mathcal{A}_n$ is a collection of basic open sets

A basic open set is a product.

$V = \prod_\xi T_\xi$ where $T_\xi = X_\xi$ for all but finitely many ξ.

Let "support" $\operatorname{supp} V = \{\xi : T_\xi \neq X_\xi\}$

since $\prod_\xi X_\xi$ is R-ccc → $\mathcal{A}_n$ is ctbl so $S_n = \bigcup \{\operatorname{supp} V : V \in \mathcal{A}_n\}$ is ctbl. So $S = \bigcup_n S_n$ is also ctbl.

So $\prod_{\xi \in S} X_\xi$ is R-ccc.

Note $\pi_S(\mathcal{A}_n) = \{\pi_S[A] : A \in \mathcal{A}_n\}$ is a max anti-chain in $\prod_{\xi \in S} X_\xi$ So by R-ccc we can find $A_n \in \mathcal{A}_n$ $\forall n$ with $\bigcup_n \pi_S(A_n)$ dense in $\prod_{\xi \in S} X_\xi$ and $\bigcup_n A_n$ is dense $\boxtimes$

Questions

Thm: If there is a Suslin line, then there is an $\mathbb{R}$-ccc space S with $S \times S$ not ccc.

Q1: Can we take S a suslin line? True Yes!

Q2: Can we have $\mathbb{R}$-ccc spaces X, Y with $X \times Y$ ccc but not $\mathbb{R}$-ccc?

Q3: If all finite products of $\mathbb{R}$-ccc spaces in a family $\mathcal{F}$ are $\mathbb{R}$-ccc does the same hold for ctbl products?

Q4: Does MA $\Rightarrow$ preservation of $\mathbb{R}$-ccc under (fin.) products?

xistence is indep ZFC.

Suslin line: is a linearly ordered set $(X, <)$ which with the order-topology is ccc but not separable.

Their existence is equivalent to the existence of Suslin trees T. These trees are of height ω_1 with ctbl levels w/o uncountable branches & w/o unctbl antichains

(First assume S is "well pruned" no terminal nodes or ones that die

The wedge topology on S is defined by letting $V_p = \{s : p \leq s\}$ for $p \in S$

For F finite $\subseteq S$ $W_F = \bigcup_{p \in F} V_p$

Take basic nbhds the sets

$V_p \setminus W_F$ F finite $\subseteq$ immediate sucessors of p

S

$\left((V_p \setminus V_{s(p)}) \times V_{s(p)} \right)_{p \in S}$

(where s(p) is an immediate sucessor of p for each $p \in S$)

are disjoint. So $S \times S$ is not ccc.

But S is R-ccc. (not too hard)

(we can also look at games.)