03/16/01
OC

(Andrés Caicedo speaks)

Thm 0: Assume κ is measurable (and GCH holds below κ). Then there is a forcing extension where $\kappa = \mathfrak{c}$ is real-valued measurable and there is a Σ^2_2 wellordering of $\mathbb{R}$.

Def: (Ulam) • κ is real-valued measurable ($RVM(\kappa)$) iff there is a κ-additive probability measure ν defined on all subsets of κ, vanishing on singletons. We call ν a witnessing probability.
• κ is 2-valued measurable $\iff$ κ is measurable (usual sense)
• κ is atomlessly measurable $\iff$ ν can be taken to be non-atomic
(ie $\forall A.\ \nu(A)>0 \Rightarrow \exists B \subseteq A$ s.t. $\nu(A) > \nu(B) > 0$).

Thm 1: (Ulam) (A) $RVM(\kappa) \Rightarrow \kappa$ is either 2-valued or atomlessly measurable.
(B) If κ is atomlessly measurable, then $\kappa \leq \mathfrak{c}$ is regular.

Proof:
(A) Trivial.
Say ν is a witness.
Case 1: ν has no atoms. Then done.
Case 2: ν has an atom A. (ie $\nu(A)>0$, but $B \subseteq A \Rightarrow \nu(B)=0$ or $\nu(B)=\nu(A)$).
Then $\mathcal{U} = \{X \subseteq \kappa \mid \nu(A \setminus X) = 0\}$ is a κ-complete ultrafilter on κ. $\square$(A)

(B) Suppose κ is atomlessly measurable, fix ν witness.
κ is regular:
ignore incompleteness complications
Def: For ν a measure, $\mathcal{N}_\nu \triangleq \{X \mid \nu X = 0\}$ = ideal of ν-null sets.
Notice $add(\mathcal{N}_\nu)$ is a regular cardinal.
But $\kappa \leq add(\mathcal{N}_\kappa) \leq \kappa$, so κ is regular.

$\kappa \leq \mathfrak{c}$:

Lemma: If ν is atomless, then $\forall A \exists B \subseteq A$ s.t. $\nu B = \frac{\nu A}{2}$.
Proof:
Start with $A_0 = A$. Define a sequence A_α as long as possible:
s.t. $\nu(A_\alpha) \geq \frac{1}{2}\nu(A)$ all α
$\nu(A_{\alpha+1}) < \nu(A_\alpha)$, $A_{\alpha+1} \subseteq A_\alpha$.
$A_\lambda = \bigcap_{\alpha<\lambda} A_\alpha$ for λ limit.
If $\forall \alpha < \omega_1$, A_α is defined, $\{A_\alpha \setminus A_{\alpha+1} \mid \alpha < \omega_1\}$ is a disjoint collection of sets of positive measure $\lightning$.

03/16 (cont.)

Sublemma: $\forall A.\ \nu(A)>0\ \forall \varepsilon>0\ \exists B\subseteq A.\ 0<\nu(B)<\varepsilon.$ $\boxtimes$ (easy split, etc...)

Now, starting with $\kappa = X_0$, define X_s for $s\in 2^{<\omega}$:

- $X_s = X_{s^\frown 0} \amalg X_{s^\frown 1}$
- $\nu(X_{s^\frown i}) = \frac{1}{2}\nu(X_s)$. (using Lemma).

For $f\in 2^\omega$, let $X_f = \bigcap_n X_{f\restriction n}$.

Then $\nu(X_f)=0$ for all f.

But $\kappa = \bigcup_f X_f$. So ν is **not** $\mathfrak{c}^+$-additive. $\blacksquare$

<u>Rmk</u>: In fact, if $\kappa\le\mathfrak{c}$ is R.V.M., then κ is weakly inaccessible, in fact weakly Mahlo, the κ^{th} weakly Mahlo, etc...

<u>Cor. of Proof</u>: N_ν is $\aleph_1$-saturated.

<u>Theorem 2</u> (Solovay?): $RVM(\kappa) \iff$

$\exists\lambda\ge\omega$ s.t. $V^{\mathbb{B}_\lambda}\models \exists j: V \xrightarrow{\equiv} N$, $\mathrm{crit}(j)=\kappa$

where $\mathbb{B}_\lambda$ is the forcing for adding λ-many random reals.

Specifically, $\mathbb{B}_\lambda$ = Borel subsets of 2^λ modulo null sets

where the measure φ is defined as follows:

If $C \triangleq \{x\in 2^\lambda \mid x\restriction J = z\}$ for J finite, $z\in 2^J$, $\varphi(C)=2^{-|J|}$, etc. — call C a <u>cylinder</u>

$\varphi(B)\triangleq \inf\{\Sigma_n \varphi(C_n) \mid B\subseteq \bigcup_n C_n,\ C_n \text{ cylinder}\}$ for Borel B.

<u>Facts</u>:

- $\mathbb{B}_\lambda$ is ccc for all λ.
- There is a "probability measure" $\nu:\mathbb{B}_\lambda\to[0,1]$, $[X]_N \mapsto \varphi(X)$.

  ($(\mathbb{B}_\lambda,\nu)$ is a <u>measure algebra</u>:
  - $\mathbb{B}_\lambda$ is complete
  - $\nu(a)=0 \iff a=0$
  - $\nu(\mathbb{1})=1$
  - ν is σ-additive ($\nu(\Sigma_n^{\mathbb{B}_\lambda} a_n) = \Sigma_n \nu(a_n)$ when $a_m\cdot a_n = 0$)
- Ex. Given any probability space $(X,\mathcal{P},\mu)$

  $\mathcal{P}/N_\mu$ is a measure algebra.

<u>Fact</u>: Any measure algebra is isomorphic to one as above. (as a measure algebra).

03/16 (cont.)

Def: • For $\mathbb{B}$ a Boolean algebra, let $\tau(\mathbb{B}) \triangleq \min\{|X| \mid X$ generates $\mathbb{B}$ (as a complete algebra)$\}$
("generating #")

• Say $\mathbb{B}$ is τ-homogeneous iff $\tau(\mathbb{B}) = \tau(\mathbb{B}\restriction a)$ $\forall a \neq \mathbb{0}$.

Facts: • $\mathbb{B}_\lambda$ is τ-homogeneous, $\tau(\mathbb{B}_\lambda) = \lambda$. (wmk|x)

• If $\mathbb{B}$ is τ-homogeneous, it is homogeneous (in the foregoing sense)

Thm 3 (Maharam): If $\mathbb{B}$ is a complete homogeneous measure algebra, then (as a measure algebra) it is isomorphic to exactly one $\mathbb{B}_\lambda$.

Fact: If $\mathbb{B} \lessdot \mathbb{B}_\lambda$ (i.e. $\mathbb{B}$ is a complete subalgebra of $\mathbb{B}_\lambda$) then $\mathbb{B} \cong \mathbb{B}_\gamma$ for some γ.

(**Rmk**: This is false for the product of ω_1 Cohen reals.)

Proof of (Solovay's) Thm:

($\Rightarrow$): Suppose $RVM(\kappa)$. Let ν be a witness.
Since N_ν is $\aleph_1$-saturated, $\mathcal{P}(\kappa)/N_\nu$ is complete, and WMA $\mathcal{P}(\kappa)/N_\nu \cong \mathbb{B}_\lambda$ for some λ.
But by saturation of N_ν, forcing with N_ν adds generic G so that

$$j: V \to N \cong Ult(V,G) \subseteq V[G]$$

well founded, $crit(j) = \kappa$.

$$\text{So } \Vdash_{\mathbb{B}_\lambda} \exists j: V \xrightarrow{\sim} N \ (crit(j) = \kappa).$$

($\Leftarrow$). Suppose $V^{\mathbb{B}_\lambda} \models \exists j: V \xrightarrow{\sim} N$, $crit(j) = \kappa$.
Let $\varphi: \mathbb{B}_\lambda \to [0,1]$ be the "prob. measure"
We want, in V, to define a measure on subsets of κ.
Let $A \subseteq \kappa$, then $\nu(A) \triangleq \varphi[\![\kappa \in j(A)]\!]$.
So $\nu: \mathcal{P}(\kappa) \to [0,1]$.
We verify ν is as wanted. ✓

$\boxtimes$

03/16 (cont)

Rmk: So if $\diamond$ is RVM, ν is a witness and $\mathcal{P}(\kappa)/\mathcal{N}_\nu$ is homog.
Then $\mathcal{P}(\kappa)/\mathcal{N}_\nu \cong \mathbb{B}_\lambda$ some λ.
Gitik-Shelah have shown $\lambda = 2^\kappa$.

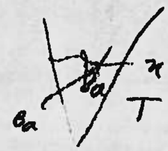
Corollary: (Silver) $RVM(\kappa) \Rightarrow$ there are no κ-Aronszajn trees.

Proof:

First, κ is weakly inaccessible. (look at $j : V \to N$ in $V^{\mathbb{B}_\lambda}$.
If $\kappa = \rho^+$, then in N, $j(\kappa) = j(\rho)^+ > \rho^+$, then we collapse cardinals, but $\mathbb{B}_\lambda$ is ccc.)

Work in $V^{\mathbb{B}_\lambda}$. Let T be a κ-tree in V, $T \subseteq \kappa$.
Then $j(T) \upharpoonright \kappa = T$. Let $\mu = j(\nu)$; $\frac{\text{in } N}{\mu}$ is a $j(\kappa)$-additive nontrivial measure on $j(\kappa)$.

For $a \in T$, let $B_a \triangleq \{ b \in T \mid a <_T b \}$.
For $c \in j(T)$, let $A_c \triangleq \{ b \in j(T) \mid c <_{j(T)} b \}$.



Then $\mu(T) = 0$, and μ is $j(\kappa)$-additive, so for some $\alpha < j(\kappa)$, $\alpha \in j(T)_\kappa$, $\mu(A_\alpha) > 0$.
Look at $b = \{ \beta \mid \beta <_T \alpha \}$.
We argue $b \in V$.

Because: say $b = \{ b_\tau \mid \tau \in \kappa \}$,
$\langle A_{b_\tau} \rangle_\tau$ is decreasing, $\mu(A_{b_\tau})$ is decreasing,
Since $\kappa \geq \omega_1$, $\exists \tau \in \kappa, \forall \rho > \tau . (\mu(A_{b_\tau}) = \mu(A_{b_\rho}).)$

Then $\mu(A_{b_\tau}) = j(\nu(B_{b_\tau}))$ and

if $c >_T b_\tau$, then either $c \in b$, $\nu(B_c) = \nu(B_{b_\tau})$
or $\nu(B_c) = 0$.

So $b = \{ c <_T b_\tau \} \cup \{ c >_T b_\tau \mid \nu(B_c) = \nu(B_{b_\tau}) \} \in V$.

$\boxtimes$

Cor: (Solovay) $RVM(\kappa) \Rightarrow \forall \lambda . V^{\mathbb{B}_\lambda} \models RVM(\kappa)$.

Proof:

Given λ, let ρ be such that $V^{\mathbb{B}_\rho} \models \exists j : V \xrightarrow{\prec} N$ $crit(j) = \kappa$.
Let $\chi > \lambda, \rho, j(\lambda)$ (for all possible values of $j(\lambda)$).
Then $V^{\mathbb{B}_\chi} \models \exists j : V \xrightarrow{\prec} N$, $crit(j) = \kappa$.
(Remember $\mathbb{B}_\rho \lessdot \mathbb{B}_\chi$)

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But $\mathbb{B}_{j(\lambda)} = \mathbb{B}_\lambda * \dot{\mathbb{B}}_{j(\lambda)}/\mathbb{B}_\lambda$

But all quotients of $\mathbb{B}_\lambda$ are of the form $\mathbb{B}_\beta$.

$$V^{\mathbb{B}_\lambda} \models \Vdash_{\dot{\mathbb{B}}_\beta} \exists\, j : V \xrightarrow{\prec} N, \quad \mathrm{crit}(j) = \kappa.$$

The generic G added by $\mathbb{B}_{j(\lambda)}$ can be seen as a list $\langle x_\alpha \mid \alpha \in j(\lambda) \rangle$ of random reals ....

In $V^{\mathbb{B}_\lambda * \dot{\mathbb{B}}_\beta}$, define $j : V[\langle x_\alpha \rangle_{\alpha\in\lambda}] \to N[\langle x_\alpha \rangle_{\alpha \in j(\lambda)}]$
by $j(\tau_{G_\lambda}) = j(\tau)_G$.

Thus $V^{\mathbb{B}_\lambda * \dot{\mathbb{B}}_\beta} \models \exists\, j : V^{\mathbb{B}_\lambda} \to N^{j(\mathbb{B}_\lambda)}$

<u>Fact</u>: Suppose $\varphi(x,y,z)$ is a formula, $t \in \mathbb{R}$, and $x < y$ is defined by $x<y \iff L(\mathbb{R}) \models \varphi[x,y,t]$.
If κ is RVM, then $<$ is <u>not</u> a wellorder.
<u>Sketch</u>:
If in $V^{\mathbb{B}_\lambda}$, $j : V \to N$, then in N, we added random reals. Looking at the random reals, since random is homogen., $j(<)$ is not a wellorder in N. $\boxtimes$

<u>So</u> if in the presence of RVM(κ) we want to define a w.o. of $\mathbb{R}$ "in an easy way", nothing much better than Σ^2_1 is possible.

<u>Fact</u>: Let φ be a Σ^2_1-formula. Then there are ψ, Υ such that

$$\varphi(x,y,\vec{z}) \iff \exists M \models \mathrm{ZFC}^{-\epsilon},\ M \text{ transitive},\ \mathbb{R} \subseteq M,\ |M| = \kappa,\ M \models \psi(x,y,\vec{z})$$
$$\iff \exists N \models \mathrm{ZFC}^{-\epsilon},\ N \text{ transitive},\ {}^{\omega}N \subseteq N,\ |N| = \kappa,\ N \models \Upsilon(x,y,\vec{z}).$$

03/16 (sev)

We are going to give a model with a Σ^2_2 defn of a w.o. of $\mathbb{R}$, of the form:

$$x \lhd y \iff \exists N \models ZFC^{-\varepsilon},\ \mathbb{R} \subseteq N,\ \text{transitive},\ N \models \psi(x,y)$$
and "some closure properties" hold for N.

(Andrés continues)

04/06/01

Recall:

Def: κ is RVM iff $\exists$ non-trivial κ-additive prob. measure ν on $\mathcal{P}(\kappa)$.

Thm: (Solovay) $RVM(\kappa) \iff \exists \lambda \geq \omega$ s.t. $V^{\mathbb{B}_\lambda} \models [\exists j : V \xrightarrow{\prec} N \ \ crit(j) = \kappa]$
($\mathbb{B}_\lambda$ forcing for adding λ random reals)

Pf: $(\Rightarrow)$ Let $\mathcal{N}_\nu$ the null ideal, then (WMA) $\mathbb{B}_\lambda \cong \mathcal{P}(\kappa)/\mathcal{N}_\nu$.
Let G be generic for forcing with $\mathcal{N}_\nu$.

Claim: $\mathbb{R}^N = \mathbb{R}^{V[G]}$, $(N = ult(V,G))$.

Pf: Let $\langle b_n \in V \rangle_{n\in\omega} \in V[G]$.
Then $\langle j(b_n)\rangle_n \in N$, (which clearly gives result).
To see that it holds, let $\langle \dot{b}_n \rangle_{n\in\omega} \in V$ be a name for the sequence.
For each n, let A_n an maximal antichain, (so $|A_n| \leq \aleph_0$).
Say $A_n = \{ a^n_\eta \mid \eta \text{ possible value}\}$, $a^n_\eta \Vdash \dot{b} = \check{\eta}$.
WMA each $a^n_\eta \subseteq \kappa$, and $\eta \neq \zeta \Rightarrow a^n_\eta \cap a^n_\zeta = \phi$.
Let $f_n : \kappa \to V$ be given by
$f_n(\beta) = $ unique η s.t. $\beta \in a^n_\eta$
Then f_n is defined a.e. and $a^n_\eta \Vdash [f_n] = [c_\eta]$
So $\Vdash [f_n] = [c_{\dot{b}_n}]$.
So $\langle [f_n]\rangle_n = \langle [c_{b_n}]\rangle_n = \langle j(b_n)\rangle_n$,
but $\langle [f_n]\rangle_n = j(\langle f_n\rangle_n)[id]$. $\boxtimes$ claim

(So in fact ${}^\omega N \subseteq N$:
if $\langle x_n\rangle_n \subseteq N$ (in $V[G]$),
then for n, let $g_n \in V$ s.t. $(g_n)_n \in V$ and $x_n = [g_n]$.
Then $\langle j(g_n)\rangle \in N$ and thus $\langle j(g_n)\rangle[id] = \langle x_n\rangle_n \in N$ $\boxtimes$)

Corollary: If φ is a fmla, $\vec{z} \in L(\mathbb{R})$, κ is RVM, and $\lhd$ is def. on $\mathbb{R}$ by:
$$x \lhd y \iff L(\mathbb{R}) \models \varphi[x,y,\vec{z}].$$
Then $\lhd$ is not a wellorder of $\mathbb{R}$.

Pf: O.w. go to $V[G]$, we have $j : V \xrightarrow{\prec} N$ $crit(j) = \kappa$, and $\mathbb{R}^N = \mathbb{R}^{V[G]}$. By elementarity, "$L(\mathbb{R}^{V[G]}) \models \varphi(\cdot,\cdot, j(\vec{z}))$" wellorders $\mathbb{R}^{V[G]}$, but G is generic for $\mathbb{B}_\lambda$ (some λ) which is homogeneous, so this cannot happen. $\boxtimes$ cor
(WMA $j(\vec{z}) = \vec{z}$.)

04/06/01 (cont.)

"(So Σ^2_1 seems the weakest possible candidate for the complexity of a w.o. of $\mathbb{R}$ if $\mathfrak{c}$ is RVM.)"
("otherwise you're just splitting hairs")

Recall: φ is Σ^2_1 iff φ is equiv. to a formula of the form $\exists M$ transitive, $\overline{\overline{M}} = \mathfrak{c}$, $\mathbb{R} \subseteq M$, $M \models \psi$.

Sup. we want φ to define a wellorder, (so we want "$M \models \psi$" to def. w.o.)
But if M changes, "$M \models \psi$" may change its meaning.

So we would like to add restrictions on what M is, so that the meaning does not change.

Thm 0: Let κ be measurable. Then there is a forcing extn where $\kappa = \mathfrak{c}$ is RVM and there is a Σ^2_2 w.o. of $\mathbb{R}$.

Proof: (WMA GCH holds below κ).
Let $\mathbb{Q} = \mathbb{B}_\kappa$. Let $\mathbb{P}$ be the Easton support product of $\mathbb{P}_\delta$, δ inaccessible.
Each $\mathbb{P}_\delta$ is the inverse product $(n<\omega)$ of $\mathrm{Add}(\delta^{+1+2n}, \delta^{+3+2n})$

We are going to look for an inner model of $V^{\mathbb{P}\times\mathbb{Q}}$

(We intend eventually to code each real by an inaccessible δ and the sequence of generics for $\mathrm{Add}(\delta^{+1+2n}, \delta^{+3+2n})$, $n<\omega$).

Let $j : V \xrightarrow{\sim} N$ be given by a normal u.f. on κ.

Claim 1: $\mathbb{P}$ preserves measurability of κ.
In fact, there is $G^* \in V$ s.t. for any $G_{\mathbb{P}}$, $\mathbb{P}$-gen/V, $G_{\mathbb{P}} \times G^*$ is $j(\mathbb{P})$-generic over N and j lifts to $j : V[G_{\mathbb{P}}] \to N[G_{\mathbb{P}}][G^*]$.

Proof:
By elementarity, $j(\mathbb{P})$ is the Easton product of $\mathbb{P} \times \underbrace{\Pi\{\mathbb{P}_\lambda \mid \lambda \in [\kappa, j(\kappa)) \text{ inaccess}\}}_{\mathbb{P}_{tail}}$ in N.
$\mathbb{P}_{tail}$ is κ^+-closed in N, and
$$|\mathcal{P}(\mathbb{P}_{tail})^N| = |(2^{j(\kappa)})^N| = |j(2^\kappa)^N| \leq (2^\kappa)^\kappa = \kappa^+,$$
So (in V) there are only κ^+-many dense subsets of $\mathbb{P}_{tail}$ which belong to N. But κ^+-closed, so define G^* as the filter generated by a decreasing sequence which meets each dense set.

04/06 (cont) To finish, it suffices to see $G_{\mathbb{P}}$ is $\mathbb{P}$-generic over $N[G^*]$.
But $N[G^*] \subseteq V$.
Now lift j as usual:

$$j(\tau_{G_{\mathbb{P}}}) = j(\tau)_{G_{\mathbb{P}} * G^*}$$

$\boxtimes$

Since $\mathbb{P}$ is ω_1-closed, $\mathbb{R}^{V[G_{\mathbb{Q}}]} = \mathbb{R}^{V[G_{\mathbb{P}}][G_{\mathbb{Q}}]}$, where $G_{\mathbb{Q}}$ is $\mathbb{Q}$-gen.

Let $A \subseteq \kappa$ code a w.o. of $\mathbb{R}^{V[G_{\mathbb{Q}}]}$ (in $V[G_{\mathbb{Q}}]$),
Let (in V) $(\delta_\alpha \mid \alpha \in \kappa)$ be inaccessibles $< \kappa$ in increasing order.
Let (in $V[G_{\mathbb{Q}}][G_{\mathbb{P}}]$) g be $\prod_\alpha G_\alpha$, where
G_α is defined from $G_{\mathbb{P}} \upharpoonright [\delta_\alpha, \delta_\alpha^{+\omega})$ given by
$n \in r_\alpha \Rightarrow g$ includes the $G_{\mathbb{P}_{\delta_\alpha^{+n+2n}}}$ generic.

<u>Claim 2</u>: κ is RVM in $V[G_{\mathbb{Q}}][g]$.
<u>Pf</u>: we need, for some λ, to find in
$V[G_{\mathbb{Q}}][g]^{\mathbb{B}_\lambda}$ an elementary embedding

$$\tilde{j} : V[G_{\mathbb{Q}}][g] \xrightarrow{\prec} M, \quad \mathrm{crit}(\tilde{j}) = \kappa.$$

Look at $j: V \longrightarrow N$.
In N, look at $j(\mathbb{Q})/\mathbb{Q}$. This is $\cong \mathbb{B}_{j(\kappa)}$.

Let H be generic for $V[G_{\mathbb{P}}][G_{\mathbb{Q}}]$ for $?$.
We have $G_{\mathbb{Q}} * H$ is N-generic for $j(\mathbb{Q})$.
If we can find a $j(g)$ in $V[G_{\mathbb{Q}}][g][H]$ which "plays in $N[G_{\mathbb{Q}} * H]$ the role of g" we are done.
(Since κ is measurable in $V[G_{\mathbb{P}}]$, in $V[G_{\mathbb{P}}][G_{\mathbb{Q}}][H]$ we have an embedding

$$j: V[G_{\mathbb{P}}][G_{\mathbb{Q}}] \longrightarrow N[G_{\mathbb{P}}][G^*][G_{\mathbb{Q}}][H]$$

~~(as κ is RVM in $V[G_{\mathbb{P}}][G_{\mathbb{Q}}]$~~ and claim 1)
We want to see $j \upharpoonright V[G_{\mathbb{Q}}][g]$ is definable in $V[G_{\mathbb{Q}}][g][H]$.

04/06 (cont.) We know what $\tilde{j}(g)$ is: $\tilde{j}(g) = g^\frown g_{tail}$,
where g_{tail} is defined from G^* (as g was) ← not all of $G_{\mathbb{P}} * G^* = \tilde{j}(G_{\mathbb{P}})$.
and $\tilde{j}(A)$.
(since $j\restriction_{V[G_0]}: V[G_0] \to N[G_0][H]$, $\tilde{j}(A) = j\restriction_{V[G_0]}(A)$).

$\boxtimes$ (κ is RVM)

Now all that is left is to see A (or rather the w.o. given by A) can be computed in $V[G_0][g]$ in a Σ^2_2 way.

<u>Def</u>: Let λ be regular. The <u>club base number</u> for λ is $\min\{\overline{\overline{X}} \mid X \subseteq \mathcal{P}(\lambda)$ s.t. $\forall$ club $C \subseteq \lambda\ \exists D \in X\ (D \subseteq C)\} \stackrel{\Delta}{=} \mathfrak{d}_\lambda$.

So by GCH, in V, $\mathfrak{d}_\lambda = \lambda^+$ for all $\lambda < \kappa$ reg.

For some λ, in $V[G_0][g]$, we added λ^{++} Cohen subsets, and their closures are club subsets of λ with no club subset in V.
So by mutual genericity, $\mathfrak{d}_\lambda^{V[G_0][g]} = \lambda^{++}$, (and λ^+ for other λ)
So now we define the w.o.:

[$x \triangleleft y$ <u>iff</u>
$\exists M$ transitive, $\mathbb{R} \subseteq M$, $\overline{\overline{M}} = \kappa$ s.t. $\mathcal{P}_{<\kappa}(M) \subseteq M$ and in M x appears before y in the coding given by $\mathfrak{d}_\lambda$.]

$\boxtimes$

(Note this is actually weaker than Σ^2_2)

<u>Thm</u>: (Woodin) If $V = L[\mu]$, then there is a forcing extension where the measurable of V is κ and RVM and there is a Σ^2_1 w.o. of $\mathbb{R}$.
(similar argument).

<u>Question</u>: Can we improve (in general) the w.o. to Σ^2_1?

07/06
(201)

<u>Thm</u>: (Kunen ??) Let κ be measurable. There is a forcing extension of V where κ is not measurable but measurability is resurrected by a further forcing which does not add any new subsets to κ.

(Fitik has an argument for this result where at the end, $\kappa = \angle \cap RVM$; at least when $V = L[\mu]$).