Forcing Axioms and Inner Models of GCH

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Abstract

Assume PFA($\mathfrak{c}$), and let M be an inner model of GCH. We investigate the consequences of assuming that $\aleph_2^V$ is a successor cardinal in M. This is believed to be impossible and, in particular, relates to a well-known conjecture due to Cummings on collapsing successors of singulars.

ZFC is the standard system of (first-order) axioms for Set Theory. It captures part of what is true in V, the universe of sets, but misses quite a bit.

Large cardinal axioms are part of what it misses, but they are not all there is, either.

Forcing axioms have been proposed as part of what they miss. It is not clear that they must be *true*, but they are very *natural*.

Here I present some conjectures which are really part of a larger project intended to understand the structure of strong forcing axioms.

Part of the intuition behind forcing axioms is the idea that the universe of sets should be "saturated enough" in the sense that it is reasonable to expect that many sets which is *possible* to have should already exist.

For example:

- $\mathsf{MA}_{\omega_1}(\sigma\text{-closed})$.

This is perhaps the weakest possible forcing axiom, because it is *true*.

A forcing $\mathbb{P}$ is σ-closed iff every countable descending sequence of conditions admits a lower bound.

If $\mathcal{D}$ is a family of ω_1-many dense subsets of $\mathbb{P}$, it is clear how to construct a descending ω_1 sequence of conditions meeting all of them. They generate a $\mathcal{D}$-*generic filter* $\mathcal{G}$.

This is all that is claimed by $\mathsf{MA}_{\omega_1}(\sigma\text{-closed})$.

Most forcing axioms are generalizations of this principle.

- For Γ a class of forcings, $\mathsf{MA}_{\omega_1}(\Gamma)$ asserts that whenever $\mathbb{P} \in \Gamma$ and $\mathcal{D}$ is a family of at most ω_1-many dense subsets of $\mathbb{P}$, there is a $\mathcal{D}$-generic filter $\mathcal{G}$.
- $\mathsf{MA}_{\omega_1}(\text{ccc})$ is simply called MA_{ω_1}.
- $\mathsf{MA} \Leftrightarrow \mathsf{MA}_{<\mathfrak{c}}$ is Martin's axiom.
- $\mathsf{MA}_{\mathfrak{c}}$ is *false*.

Historically, it was the first (non-provable) forcing axiom to be considered. It was introduced by Martin.

Theorem 1 (Martin, Solovay). MA *implies all* $\underset{\sim}{\Sigma}^1_2$*-sets of reals are Lebesgue measurable.*

At the other end of the spectrum, we have:

- $\mathsf{PFA} \Leftrightarrow \mathsf{MA}_{\omega_1}$(proper).

MA_{ω_2}(proper) is *false*. In fact, Todorcěvić has shown that PFA implies $\mathfrak{c} = \aleph_2 = 2^{\aleph_1}$.

A forcing is proper iff it preserves stationary sets of $\mathcal{P}_{\omega_1}(S)$ for any S.

PFA was introduced by Baumgartner and Shelah.

- SPFA $\Leftrightarrow$ MA_{ω_1}(semiproper).

A forcing $\mathbb{P}$ is semiproper iff given any η sufficiently large for stationarily many $X \in \mathcal{P}_{\omega_1}(V_\eta)$ it is the case that $X \prec V_\eta$, $\mathbb{P} \in X$, and $\omega_1 \cap X = \omega_1 \cap X[G]$.

SPFA was introduced by Foreman, Magidor and Shelah.

- $\mathsf{MM} \Leftrightarrow \mathfrak{c} = \mathsf{MA}_{\omega_1}$ (stationary set preserving).

$\mathbb{P}$ is stationary set preserving iff it preserves stationary subsets of ω_1.

MM, Martin's Maximum, was also introduced by Foreman, Magidor and Shelah.

- $\mathsf{MA}_{\omega_1}(\mathbb{P})$ is *false* if $\mathbb{P}$ is not stationary set preserving.

So MM is the strongest forcing axiom there is.

Foreman, Magidor and Shelah showed:

Theorem 2. SPFA *implies* MM, *so they are equivalent.*

- MA, PFA and SPFA can all be forced.
- MA $\Leftrightarrow$ MA(ccc of size at most continuum).

The forcing proof of the consistency of MA is very natural, only some bookkeeping is required: You want to have sufficiently generic filters for all ccc forcings. For these, you only need to *add* those generics by forcing. So you just iterate, adding them "one by one." Since only ccc posets of size at most continuum need be considered, this is easy to achieve, modulo some bookkeeping.

There is no such *reflection* result for PFA. The proof of its consistency uses the same idea: Just iterate adding generics for proper posets. The problem is when to stop.

Also, some large cardinal strength is now necessary. For example, Todorcěvić showed that PFA implies $\neg\Box_\kappa$ for any $\kappa \geq \omega_1$.

Supercompact cardinals provide the required amount of reflection.

κ is supercompact iff for all λ there is an elementary embedding

$$j : V \to M$$

with critical point κ and such that

- $\lambda < j(\kappa)$.
- ${}^{\lambda}M \subseteq M$.

Supercompact cardinals also provide the necessary bookkeeping device:

Theorem 3 (Laver). *Let κ be supercompact. Then there is $\ell : \kappa \to V_\kappa$ such that for every x and every λ sufficiently large there is a λ-supercompact embedding $j : V \to M$ such that $j(\ell)(\kappa) = x$.*

ℓ is an oracle that can be used to anticipate any set in an inductive construction.

SPFA can be proved by essentially the same argument.

In a sense, this is a problem, because it is the only known way to force these axioms. It is the only known way to force *most* forcing axioms.

There are many natural intermediate axioms as well. Let me mention some:

- SPFA($\mathfrak{c}$). We restrict to forcings of size at most $\mathfrak{c}$.
- MM($\mathfrak{c}$).

Recall that SPFA and MM are equivalent. However, MM($\mathfrak{c}$) and SPFA($\mathfrak{c}$) are not.

Theorem 4 (Folklore?). *A strong cardinal suffices to force* SPFA($\mathfrak{c}$).

This is obtained by running the standard argument, using an appropriate version of Laver's theorem for strong cardinals.

Theorem 5 (Woodin). MM($\mathfrak{c}$) *implies Projective Determinacy.*

Woodin has obtained models of MM($\mathfrak{c}$) by nonstandard methods. However, he is required to start from models of determinacy.

- BMM: $H_{\omega_2} \prec_{\Sigma_1} H_{\omega_2}^{V^{\mathbb{P}}}$ for all stationary set preserving forcings $\mathbb{P}$.
- BSPFA.

Bounded axioms were introduced by Goldstern and Shelah. A Σ_1-reflecting cardinal suffices to prove the consistency of BSPFA. Again, using the standard method.

Theorem 6 (Asperó, Welch). BSPFA *does not imply* BMM.

This result is obtained by forcing BSPFA via the usual argument, starting from a small large cardinal hypothesis, and showing that ψ_{AC}, a consequence of BMM in the presence of large cardinals, fails here.

Definition 7 (Woodin). ψ_{AC} holds iff given any stationary, costationary subsets of ω_1, S and T, there is a club $C \subseteq \omega_1$, an $\alpha < \omega_2$, and a bijection $\pi : \omega_1 \to \alpha$ such that

$$S \cap C = \{\beta \in C : \mathrm{ot}(\pi``\beta) \in T\}.$$

The standard method is quite flexible, and many results can be proved by slight variations of it. The problem is that starting with models of Choice no other known method has been found.

Here is a sample result, illustrating how flexible the method is:

Theorem 8. 1. *Suppose there is a strong cardinal κ. Then there is a κ-cc semiproper forcing $\mathbb{P}$ of size κ such that in $V^{\mathbb{P}}$,* SPFA($\mathfrak{c}$) *and* BSPFA *hold.*

2. *If there is a measurable above κ, then in $V^{\mathbb{P}}$ Woodin's axiom ψ_{AC} holds.*

3. *Suppose $L[\mathcal{E}]$ is a finestructural inner model with a strong cardinal and a measurable above, and such that there are no Woodin cardinals in inner models of $L[\mathcal{E}]$. Then there is a Σ^1_5-well-ordering of $\mathbb{R}$ in $L[\mathcal{E}]^{\mathbb{P}}$.*

To what extent is the standard argument a requirement?

Here is a test question:

Question 9. *Suppose* MM *holds and there are no inaccessibles. Let r be a Cohen real. Can* MM *be recovered in an outer model of $V[r]$?*

The expected answer is No.

Theorem 10. *Suppose* $M \models \mathsf{MM}$. *Suppose* $r \in \mathcal{P}(\omega_1^M) \setminus M$. *Let* N *be an outer model of* $M[r]$ *such that* $N \models \mathsf{MM}$. *Then* $\omega_2^N > \omega_3^M$.

Proof: This is a consequence of a couple of deep structural results.

Theorem 11 (Veličković). *If* MM *holds and* M *is an inner model such that* $\omega_2^M = \omega_2$, *then* $\mathcal{P}(\omega_1) \subseteq M$.

An immediate corollary is that $\omega_2^N > \omega_2^M$.

Now suppose that $\omega_2^N = \omega_3^M$.

Theorem 12 (Shelah). *Suppose $P \subseteq W$ are inner models and μ is a regular cardinal of P.*

Suppose $(\mu^+)^P$ is a cardinal of W. Then $W \models \mathrm{cf}(|\mu|) = \mathrm{cf}(\mu)$.

It follows that $\mathrm{cf}^N(\omega_2^M) = \omega_1^N$. But MM implies $2^{\omega_1} = \omega_2$, and therefore a cofinal set of levels of $(2^{<\omega_2})^M$ is a weak Kurepa tree in V, i.e., a tree of size and height ω_1 with more than ω_1-many branches.

This contradicts a result of Baumgartner. $\square$

Shelah's result also holds in some cases where μ is singular:

Theorem 13 (Shelah). *Suppose $P \subseteq W$ are inner models and μ is a cardinal of P such that $\square_\mu$ holds in P.*

Suppose $(\mu^+)^P$ is a cardinal of W. Then $W \models \mathrm{cf}(|\mu|) = \mathrm{cf}(\mu)$.

In fact, Cummings, Foreman and Magidor showed that $\square^*_\mu$ suffices.

Here is another test question:

Question 14. *Suppose* MM *holds and* M *is an inner model of* GCH. *Is* ω_2^V *inaccessible in* M*?*

The expected answer is yes.

Very little is known about this question, and it is contained in the following remarks:

Claim 15. *Assume* $\mathsf{PFA}(\mathfrak{c})$, *and let* $M \models \mathsf{GCH}$ *be an inner model. Then, without loss of generality,* $\omega_1^M = \omega_1^V$. *More carefully, there is an inner model* N *with* $M \subseteq N \subseteq V$ *such that* $N \models \mathsf{GCH}$, $\omega_1^N = \omega_1^V$, *and* ω_2^V *is inaccessible in* N *iff it is inaccessible in* M.

Proof: Let $\kappa = \omega_1^V$, and $\mathbb{P} = \texttt{Coll}(\omega, < \kappa)$. Then $\mathbb{P} \in M$ and $\mathbb{P}$ is ccc.

Lemma 16. *Suppose ω_2^V is a successor in M, say $\omega_2^V = (\lambda^+)^M$. Then λ is singular in M.*

Proof: By Shelah's result, it suffices to prove that $\mathrm{cf}(\lambda) \neq \omega_1$ in V.

Claim 17. $\mathrm{cf}(\lambda) = \omega$.

Proof: By contradiction, suppose $\mathrm{cf}(\lambda) = \omega_1$. Then a cofinal set of levels of $(2^{<\lambda})^M$ would be a weak Kurepa tree in V. Contradiction. $\triangledown$ $\triangle$

The argument splits now into two cases, according to whether ω_2^V is inaccessible in M or not.

If $\omega_2^V = (\lambda^+)^M$, then by the lemma λ is singular, thus limit, in M. In particular, it is bigger than $(\kappa^+)^M$.

Otherwise, ω_2^V is inaccessible in M.

In both cases, it is certainly bigger than $(\kappa^+)^M = |\mathcal{P}(\mathbb{P})|^M$, so $|\mathcal{P}^M(\mathbb{P})| = \aleph_1$. By $\mathsf{MA}_{\aleph_1}$ there is a $\mathbb{P}$-generic G over M. In $M[G]$, GCH holds, $\omega_1^{M[G]} = \omega_1$, and if $\omega_2^V = (\lambda^+)^M$, then λ is preserved since $\mathbb{P}$ is κ-cc in M. $\square$

The argument above, and the strengthened version of Shelah's result quoted before, in fact show the following:

Theorem 18. *Suppose M is an inner model of* GCH, PFA($\mathfrak{c}$) *holds and ω_2^V is a successor cardinal in M. Then $\omega_2^V = (\lambda^+)^M$, where $\mathrm{cf}(\lambda) = \omega$ and $\square_\lambda^*$ fails in M.*

So, if the question has a negative answer, we arrive at the following situation:

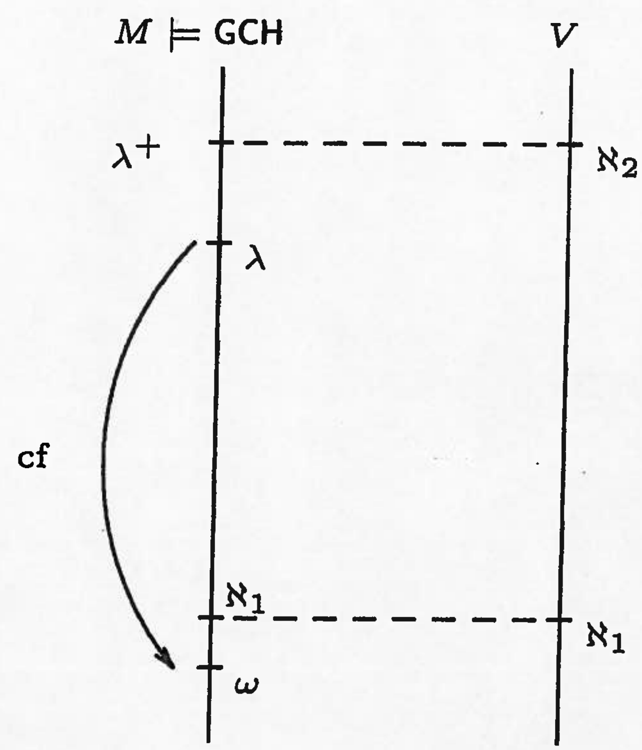
$M \models \mathsf{GCH}$ is an inner model, $\omega_1 = \omega_1^M$, $\omega_2 = (\lambda^+)^M$, where $\mathrm{cf}^V(\lambda) = \omega$.

This is conjectured to be impossible:

Conjecture 19 (Cummings). *Assume $N \subseteq W$ are inner models, μ is a cardinal of N, and $(\mu^+)^N$ is a cardinal of W. Then $W \models \mathrm{cf}(|\mu|) = \mathrm{cf}(\mu)$.*

The following is essentially contained in results of Shelah and Džamonja:

Theorem 20. *With M and λ as above, in M the approachability property fails at λ and there are no uniformly almost disjoint sequences for λ. In particular,* $\mathrm{cf}^M(\lambda) = \omega$.



$M \models \mathsf{GCH}$
V
λ^+
$\aleph_2$
λ
cf
$\aleph_1$
$\aleph_1$
ω

By results of Cummings,

Theorem 21. *V is not a weakly proper forcing extension of M, and there is no inner model of V that computes ω_2 correctly where* CH *holds.*

A forcing $\mathbb{P}$ is weakly proper iff every countable set of ordinals in $V^{\mathbb{P}}$ is covered by a countable set in V. This notion was introduced by Woodin.

In particular, no model as M can arise as an inner model of the standard forcing extension that satisfies PFA($\mathfrak{c}$).

Finally, by results of Shelah, Foreman, Magidor, and Kojman:

Theorem 22. *If $\omega_2^V = \aleph_{\omega+1}^M$, then there are no very weak squares at $\aleph_\omega$ in M, and*
$(S_{\omega_1}^{\omega_2^V})^M =_{\mathrm{NS}_{\omega_2}} S_{\omega_1}^{\omega_2}$.

All these results suggest that, if consistent, the existence of such an M must have considerable consistency strength.

Let me close with an easy observation, and another question:

Suppose $\mathsf{PFA}(\mathfrak{c})$ holds, M is an inner model of GCH that computes $\aleph_1$ correctly, and where $\aleph_2^V$ is a successor. Then there is a real r such that $M[r]$ computes $\aleph_2$ correctly. In particular, we have violated CH by adding a real.

Question 23 (Woodin). *Suppose* $\underset{\sim}{\delta}_2^1 = \aleph_2$ *and* $\forall r \in \mathbb{R}(r^\sharp$ *exists). This is implied, for example, by* MM($\mathfrak{c}$). *Let* M *be an inner model correctly computing* $\aleph_2$. *Must* CH *fail in* M*?*

Notice that under the hypothesis of the question, there are reals r such that $M[r] \models \neg\mathsf{CH}$. However, the arguments above do not apply, since now there may be weak Kurepa trees in V.