Set Theory: 30 Jan — Caicedo

Determinacy from Large Cardinals

$A \subseteq X^\omega$

Game. $G_\omega(A)$

$$\begin{array}{c|ccc} \mathrm{I} & x_0 & & x_2 \\ \hline \mathrm{II} & & x_1 & \ldots \end{array}$$

$\vec{x} = \langle x_0, x_1, \ldots \rangle$

I wins $\iff \vec{x} \in A$

else II wins

A or $G_\omega(A)$ is <u>determined</u> iff I or II has a winning strategy.

I has a winning strat: $\exists x_0 \forall x_1 \exists x_2 \forall x_3 \ldots \quad \vec{x} \in A$

II : $\forall x_0 \exists x_1 \forall x_2 \exists x_3 \ldots \quad \vec{x} \notin A$

Thm: Any finite game is determined.

$\omega = \{0, 1, \ldots\}$ discrete topology

$\omega^\omega \cong \mathbb{R} \setminus \mathbb{Q}$ (continued fraction of a real #)

(we have a Borel structure)

$(\omega^\omega, \mathcal{B}) \cong (\mathbb{R}, \mathcal{B})$

$\uparrow$ natural measure here — $\uparrow$ Lebesg measure here λ

For "nice collections" $\Gamma \subseteq \mathcal{P}(\omega^\omega)$ ~~are~~

if sets in Γ are determined
then they are measurable,
have the Baire property.
perfect (either wor 9/o)

motivation for determinacy

Martin : Borel determinacy holds

his proof demonstrated a nice way of showing this result.

(before) Gale-Stewart: Open games are determined

Martin showed Borel sets can be "mapped" to an open set in another space.

then we make sure there is a natural way to translate the game.

$L(\mathbb{R})$ is the model that starts w/ the reals and adds what we need for set theory
- Constructable from reals universe

$L_0(\mathbb{R}) = \mathbb{R}$

$L_{\alpha+1}(\mathbb{R}) = \mathrm{Def}(L_\alpha(\mathbb{R}), \in, \mathbb{R})$

limit $L_\lambda(\mathbb{R}) = \bigcup_{\alpha<\lambda} L_\alpha(\mathbb{R})$

this ~~of the~~ all of the natural sets like in Analysis

Large cardinals have to do with whether or not we can say these sets are determined.

Solovay proved: If determinacy holds in $L(\mathbb{R})$
ω_1 is measurable in $L(\mathbb{R})$

Martin proved: If "
ω_2 is measurable in $L(\mathbb{R})$

It became clear that determinacy has to do with measurability

Conj: From ..., $L(\mathbb{R})$ is determined (we should be able to prove this)

Large Cardinals

Reflection Thm:
If φ is true in V, then φ holds in V_α for arbitrary large α.

(This should be true in general though!)

L.C. are formalizations of strong reflection principles

Mostowski: $Aut(V) = \{id\}$

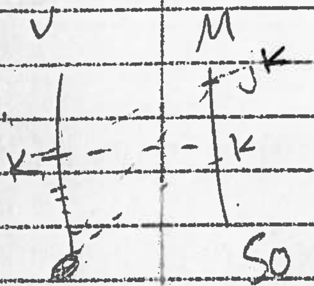
So, we go to elementary embeddings

ele. embedding

$$j : V \longrightarrow M$$

$\forall\ x_1, \dots, x_n$
$\forall\ \varphi$

$$V \models \varphi[x_1, \dots, x_n] \Leftrightarrow M \models \varphi[j(x_1), \dots, j(x_n)]$$



So we only talk about certain sets in M
measures.

$j\restriction_{ORD} : ORD \longrightarrow ORD$ order preserving
$j(\alpha) \geq \alpha \ \ \forall \alpha$

If $j \neq id$ then $j(\alpha) > \alpha$ for some α.

The first α is called the critical point of j
$cp(j) = \kappa$

We say $X \subseteq \kappa$ has measure 1, (μ_j)
iff $\kappa \in j(X)$.
(else $\mu_j(X) = 0$)

Existence of measures like this give us a way to extract these embeddings

From μ_j we can get an ele. embedding
$i : V \longrightarrow Ult(V, \mu_j)$

Large Cardinals have to do with existence of measures!

extenders

This has been a process.

$$\begin{array}{ccc} j: V & \longrightarrow & M \\ \searrow & & \uparrow F \\ & Ult(V, u_i) & \longrightarrow Ult(N, V) \\ & \| & \\ & N & \end{array}$$

So we get a sequence of measures are called extenders.

In Modern extenders (Jensen)

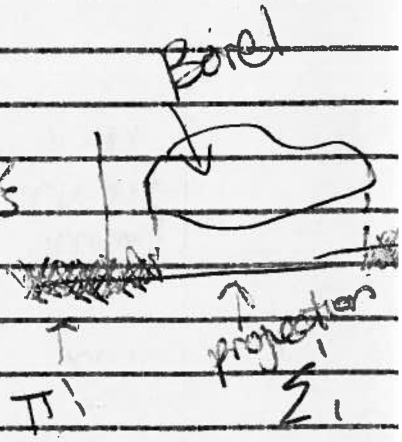
$$E: \mathcal{P}(\kappa) \to \mathcal{P}(\lambda)$$

(Martin) from these measurable cardinals sets that are Π^1_1 are determined

$A \subseteq \omega^\omega$ iff

$\exists B \subseteq \omega^\omega \times \omega^\omega$ Borel

$A = \neg p B = \{x : \forall y \ (x, y) \notin B\}$



$\Sigma_1 \quad \Pi_1 \quad \Sigma_2 \quad \Pi_2 \quad \ldots$

The proof of this is really 2 arguments.

① We can associate to A Π^1_1 a tree T (homogeneous) described in terms of measures.

with $A = p[T]$

② If T is homo and $A = p[T]$ then
A is Determined

To see if sets are determined we just need to check that they are projections of nice trees.

$A \subseteq \omega^\omega$ homo suslin:
$\exists\, \gamma\ T\quad \omega \times \gamma$ Tree.

& There are models $(M_s, f_s, j_{s,t} : s \subseteq t \in \omega^{<\omega})$ (*)
with $M_\emptyset = V$

$j_{s,t} : M_s \to M_t$ is ele.
$f_s \in j_{\emptyset,s}(T_s)$ & $s \subseteq t \Rightarrow j_{s,t}(f_s) \subseteq f_t$

given s, $T_s = \{ f : (s,f) \in T \}$
& $\forall x,\ x \in A \iff \langle M_\emptyset, M_{x \restriction 1}, \dots \rangle$ has a well founded limit

Let $x \in A$. $\quad [T] = \{ (x, F) \in \omega^\omega \times \gamma^\omega :$ branches
$\forall n\, (x \restriction n, F \restriction n) \in T \}$.

$F \restriction n \in T_{x \restriction n}$

Tree is a tree.
Suslin is the projection
homo is the well-founded limit.

Descriptive set theory:
iteration trees. "What kind of iteration trees can we produce?"

the $\subseteq$ Woodin model

We say tree is κ-homo if the measures associated to these embedding
$cp(j) \geq \kappa$ $\forall$ embedding j in $(*)$

(Martin-Steel) ← can be anything
If δ is Woodin, $A \subseteq (\omega \times \omega)$
$B = \{x \in \omega^\omega : \forall y \ \langle x, y \rangle \notin A\}$
A δ^+-hom Suslin $\Rightarrow$ B κ-hom. Sus. $\forall \kappa <$

woodin
measurable cardinals below

Π^1_2
if δ is woodin and κ is measurable.

$B = \neg p A$ $\quad A \ \Pi^1_1 \rightarrow A$ κ-hom. Sus.
$\Rightarrow B$ ρ-homo. Sus. $\forall \rho < \delta$

Π^1_3 δ'
Π^1_2 δ
Π^1_1 κ

If you have infinitely many Woodin cardinals, we have infinitely many determined sets.

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