A Brief Introduction to Forcing

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I. Statement of Purpose

Forcing is a method for establishing consistency results (for, say, combinatorial statements with respect to some first order axiomatization of set theory).

The idea is that if a model $\mathcal{M}$ of set theory does not satisfy some statement, it may be due to lack of witnesses. So forcing is a method for adding these witnesses to the model, and still preserving the fragment of set theory the model satisfies.

For example: Suppose $\mathcal{M}$ is countable.

- If $\mathcal{M} \models \neg\mathsf{CH}$, then (in V) there is a bijection between

$$\mathbb{R}^{\mathcal{M}} = \{\, y : \mathcal{M} \models y \in \mathbb{R} \,\}$$

and

$$\omega_1^{\mathcal{M}} = \{\, \tau : \mathcal{M} \models \tau \in \omega_1 \,\},$$

but no such bijection belongs to $\mathcal{M}$.

- Since $\mathcal{M}$ is countable, there are lots of reals with no representative in $\mathcal{M}$—$x \in \mathcal{M}$ is a real *in the sense of* $\mathcal{M}$ if, say,

$$\mathcal{M} \models x : \mathbb{N} \to \{0, 1\}.$$

Maybe CH fails (in V), but $\mathcal{M} \models \mathsf{CH}$, for the simple reason that not enough reals are in $\mathcal{M}$.

- If $\mathcal{M} \models \mathfrak{A}$ and $\mathfrak{B}$ are $\mathcal{L}$-structures (for some first-order language $\mathcal{L}$), and

$$\mathcal{M} \models \mathfrak{A} \equiv \mathfrak{B} \text{ but } \mathfrak{A} \not\cong \mathfrak{B},$$

it may well be that $\mathfrak{A} \cong \mathfrak{B}$, but $\mathcal{M}$ does not contain any isomorphism between them.

- Etc.

It may also be that the background theory we are working in (usually ZFC or an extension of ZFC by large cardinal axioms) simply does not decide the questions we are asking. So forcing is the method for making this explicit, hence we know that some 'theorems' *cannot* be proven.

For example:

- **Whitehead Problem:** Let G and H denote infinite abelian groups. G is *free* iff it has a linearly independent set of generators.

  G is a W-group iff for any surjective homomorphism $\phi : H \to G$ with kernel $\mathbb{Z}$ there is a homomorphism $\tau : G \to H$ such that $\phi \circ \tau = \text{id}$.

  Is every W-group free?

Shelah arrived at proper forcing by trying to solve this problem. ZFC is not enough to decide the answer.

- **Kaplansky's Conjecture:** Let $\mathfrak{X}$ be a compact Hausdorff space, and let B be a commutative Banach algebra.

  Is every homomorphism $\phi : C(\mathfrak{X}) \to B$ continuous?

Woodin showed that **ZFC** is not enough to decide the answer, not even restricting ourselves to the case $\mathfrak{X} = [0,1]$.

- Assuming the countable version of the axiom of Choice (enough for most of classical analysis) but not its full extent—so we cannot carry out the usual construction of Vitali's nonmeasurable set—, is there a nonmeasurable subset of $\mathbb{R}$?

Solovay showed that **ZFC** is not enough to know this.

Forcing is the way of getting all these results.

A partial, but useful analogue occurs in algebra. Given a field K and an irreducible polynomial $p \in K[x]$, it may happen that the equation

$$p(x) = 0$$

does not have solutions in K.

But there is a standard way of extending K so such a root is added: We consider $K[x]/(p)$.

We then define an 'interpretation' of the field operations inside this space, and an interpretation of the elements of K, so $\mathcal{K} = K[x]/(p)$ can be considered an extension of K. Also, if α is the class of x in $\mathcal{K}$, then

$$\mathcal{K} \models p(\alpha) = 0.$$

This is a good analogue (i.e., we want to copy some features of this example):

- $\mathcal{K}$ so constructed is in a precise sense minimal with respect to the properties of extending K and having roots for p.
- Due to this minimality, we know 'in K' what properties is $\mathcal{K}$ going to have (i.e, we know its first order theory with parameters).

This is because we have a way of talking 'in K' about such properties. For example, (first order) statements in $\mathcal{K}$ about α can be routinely converted into (second order) statements in K about the ideal (p).

This is a **bad** analogue:

- Most technical difficulties in set theory come from the fact that we work in a first order framework, but want to talk about second order properties.

In the context of forcing, the role of K is played by some model like $\mathcal{M}$ in our examples above, it is called the **ground model**; the role of the ideal (p) is played by the **generic filter** G, and the role of $\mathcal{K} = K(\alpha)$ is played by the **generic extension** $\mathcal{M}[G]$.

A careful analysis must be made to convert 2^{nd} order statements about the generic and the generic extensions into 1^{st} order statements about the ground model. We need such a conversion, since for all we know, we could really be working in V (not in a tiny countable model), and then the generic extensions of V would be proper classes, but our framework (ZFC) is first order, and things like quantification over proper classes are not allowed, and in general do not even make sense.

- In the case of fields, any 'minimal' extension of K is isomorphic to $\mathcal{K}$. With forcing that is never the case: If two forcing extensions are isomorphic, then they are in fact equal. Moreover, it is not even true that forcing extensions are elementarily equivalent (some technical requirement on the forcing is necessary, for example, **weak homogeneity**).

II. The Forcing Machinery

We will work in V and talk about generic extensions of V until the last section (on metamathematics).

There are two standard and equivalent approaches to forcing: via Boolean algebras and via partial orders. We adopt the latter.

Definition 1 1. *A partially ordered set (a* **notion of forcing***)* $\mathbb{P}$*—really,* $(\mathbb{P}, \leq_{\mathbb{P}}, \mathbf{1})$*, consists of*

- *A set* $\mathbb{P}$ *of* **conditions**.
- *A relation* $\leq_{\mathbb{P}}$ *on* $\mathbb{P} \times \mathbb{P}$ *which is reflexive and transitive and such that* $\mathbf{1}$ *is maximum.*

2. *Let* p *and* q *be conditions.* p **extends** q *(or is* **stronger***) iff* $p \leq q$*.*

3. p *and* q *are* **compatible** *iff there is* r*,* $r \leq p$ *and* $r \leq q$*. We write* $p \parallel q$*. Otherwise, they are incompatible, and we write* $p \perp q$*.*

We can also ask of $\leq_{\mathbb{P}}$ to be antisymmetric, if we want. In examples more elaborated than the ones we will present some simplifications arise from our more generous context. We can also eliminate the clause on 1 without any gain of generality—and, again, adding complications in some contexts.

Examples:

- $\mathbb{P} = 2^{<\omega}$, $p \leq q$ iff $p \supseteq q$.

In this case, $1 = \emptyset$. p and q are compatible iff $p \cup q$ is a function.

- $\mathbb{P} = \mathrm{Coll}(\omega_1, \mathfrak{c}) := \{ f \vdots \omega_1 \to \mathfrak{c} : |f| \leq \aleph_0 \}$, $p \leq q$ iff $p \supset q$.
- (Solovay's random forcing) $\mathbb{P} = \{ X \subset \mathbb{R} : X$ is Borel measurable and $\mu(X) > 0 \}$, $p \leq q$ iff $p \subseteq q$.

In this case, $p \parallel q$ iff $p \cap q$ has positive measure. Here, $1 = \mathbb{R}$.

Why does $p \leq q$ denote that p 'extends' q? This is the so called Western convention. The Eastern convention (mainly used by Shelah and his collaborators) uses $p \geq q$.

The reason is that in the Boolean algebra context $\leq$ coincides with the Boolean algebra ordering. And there is a canonical order-preserving embedding of notions of forcing into complete Boolean algebras (Stone representation theorem).

Definition 2 *Let $\mathbb{P}$ be a notion of forcing. A subset $\mathcal{D} \subset \mathbb{P}$ is* **dense** *iff for every $p \in \mathbb{P}$ there is $q \in \mathcal{D}$ with $q \leq p$.*

Examples:

- $\mathcal{D} = \{ f \in 2^{<\omega} : n \in \operatorname{dom} f \}$ is dense in $2^{<\omega}$.
- $\mathcal{D} = \{ f : \alpha \in \operatorname{ran} f \}$ is dense in $\mathbb{P} = \operatorname{Coll}(\omega_1, \mathfrak{c})$, for any $\alpha < \mathfrak{c}$.
- $\mathcal{D} = \{ X : X \text{ is compact and } \mu(X) > 0 \}$ is dense in Solovay's random forcing.

Definition 3 *$\mathcal{F} \subset \mathbb{P}$ is a* **filter** *iff*

- $1 \in \mathcal{F}$.
- *For every $p, q \in \mathcal{F}$, there is $r \in \mathcal{F}$, $r \leq p$, $r \leq q$.*
- *For every $p \in \mathcal{F}$, if $p \leq q$, then $q \in \mathcal{F}$.*

Example:

If $\mathcal{F}$ is a filter in $2^{<\omega}$, then $\bigcup \mathcal{F}$ is a function, since any two elements of $\mathcal{F}$ are compatible.

Definition 4 *Let $\mathfrak{D} \subset \mathcal{P}(\mathbb{P})$ be a collection of dense subsets of $\mathbb{P}$. We say $G \subset \mathbb{P}$ is* $\mathfrak{D}$**-generic** *iff G is a filter and meets all the elements of $\mathfrak{D}$. If*

$$\mathfrak{D} = \{\mathcal{D} \subset \mathbb{P} : \mathcal{D} \text{ is dense}\},$$

we say G is $\mathbb{P}$-generic (over V).

Really, the definition is relative to a given nice model $\mathcal{M}$ of set theory such that $\mathfrak{D}, \mathbb{P} \in \mathcal{M}$. Then we say, for example, that G is $\mathbb{P}$-generic over $\mathcal{M}$ iff G is a filter and meets every dense subset of $\mathbb{P}$ which belongs to $\mathcal{M}$.

Theorem 1 *(Rasiowa-Sikorsky) Let $\mathfrak{D}$ be a countable collection of dense subsets of $\mathbb{P}$. Then, for every $p \in \mathbb{P}$, there is a $\mathfrak{D}$-generic filter G with $p \in G$.*

Dem. Define $p \geq p_0 \geq p_1 \geq \ldots$ such that if

$$\mathfrak{D} = \{ \mathcal{D}_n : n \in \omega \},$$

then $p_n \in \mathcal{D}_n$. Let

$$G = \{ q \in \mathbb{P} : \exists n \, (q \geq p_n) \}. \quad \square$$

Notice that this theorem for $\mathbb{P} = 2^{<\omega}$ or $\omega^{<\omega}$ is just an instance of the Baire category theorem.

Examples:

- Let $\mathcal{Q}$ be a countable dense linear order without endpoints. Any order isomorphism $g : \mathbb{Q} \to \mathcal{Q}$ induces a $\mathfrak{D}$-generic filter for $\mathbb{P}$, where

$$\mathbb{P} = \{ f \vdots \mathbb{Q} \to \mathcal{Q} : \quad f \text{ is order preserving and } |f| < \aleph_0 \}$$

and $\mathfrak{D} = \{ \mathcal{D}_q : q \in \mathbb{Q} \} \cup \{ \mathcal{D}^r : r \in \mathcal{Q} \}$, where for each $q \in \mathbb{Q}$ and $r \in \mathcal{Q}$ we let

$$\begin{aligned} \mathcal{D}_q = &\quad \{ p \in \mathbb{P} : q \in \operatorname{dom} p \} \quad \text{and} \\ \mathcal{D}^r = &\quad \{ p \in \mathbb{P} : r \in \operatorname{ran} p \}. \end{aligned}$$

Notice that $\mathfrak{D}$ is a collection of dense sets, and $G = \{ p \subset g : |p| < \aleph_0 \}$ is $\mathfrak{D}$-generic. Conversely, for any $\mathfrak{D}$-generic filter G, $\bigcup G : \mathbb{Q} \to \mathcal{Q}$ is an isomorphism.

In general, the Rasiowa-Sikorsky theorem cannot be improved to allow for ω_1 dense subsets of $\mathbb{P}$. For example:

- Let

  $$\mathbb{P} = \mathrm{Coll}(\omega, \omega_1) = \{\, p \vdots \omega \to \omega_1 : |p| < \aleph_0 \,\}.$$

  If G is $\mathfrak{D}$-generic, where $\mathcal{D} \in \mathfrak{D}$ iff

  $$\begin{aligned} \mathcal{D} = \quad & \{\, p : n \in \operatorname{dom} p \,\} \quad (\text{some } n \in \omega), \text{ or} \\ & \{\, p : \alpha \in \operatorname{ran} p \,\} \quad (\text{some } \alpha \in \omega_1), \end{aligned}$$

  then $\bigcup G : \omega \to \omega_1$ would be onto, a contradiction.

- If $\mathbb{P}$ is Solovay's random forcing, and there are $\mathfrak{D}$-generic filters for any collection $\mathfrak{D}$ of ω_1 dense subsets of $\mathbb{P}$, then CH is false. This follows by an easy modification of the proof of the following fact:

- There are no generics for Solovay forcing $\mathbb{P}$. I.e., no filter G $\mathbb{P}$-generic over V exists (in V). This is because, if G is a counterexample, then $\bigcap G$ is nonempty, because the compact sets are dense. But, for each r,

$$\mathcal{D}_r = \{ X \in \mathbb{P} : r \notin X \}$$

is also dense, so $r \notin \bigcap G$.

- Similarly, there are no generics for $2^{<\omega}$: Let

$$\begin{aligned} \mathcal{D}_n &= \{ p : n \in \operatorname{dom} p \} \\ \mathcal{D}^f &= \{ p : p \not\subset f \} \end{aligned}$$

for each $n \in \omega$ and $f \in 2^\omega$. Since the $\mathcal{D}_n$ and the $\mathcal{D}^f$ are dense, if G is $2^{<\omega}$-generic, then $\bigcup G$ is a function with domain ω, say f; but then we get a contradiction, by considering any element of $G \cap \mathcal{D}^f$.

- As a matter of fact, for most notions of forcing $\mathbb{P}$, no $\mathbb{P}$-generics exist. To ensure this, it is enough that $\mathbb{P}$ is **non-atomic**:

$$\forall r\, \exists p_1, p_2 \leq r\, (p_1 \perp p_2).$$

There is also a characterization of those $\mathbb{P}$ for which we can prove that there is a collection of ω_1 dense subsets such that no generic meets all of them. Such $\mathbb{P}$ are not **stationary-set preserving**.

Forcing is the means to extend the universe by adding a generic subset G to it. The examples above show that in general this extension is nontrivial (i.e., we are really adding something).

Fix a notion of forcing $\mathbb{P}$. We want to define what $V[G]$ is, for G a filter $\mathbb{P}$-generic over V.

But since $G \notin V$, we introduce a language, to be able (in V) to talk about it. It is the language of set theory, augmented with constant symbols:

Definition 5 *A* **$\mathbb{P}$-name** *is a $\mathbb{P}$-name of rank $\leq \alpha$ for some ordinal α, where τ is a $\mathbb{P}$-name of rank $\leq \alpha$ iff*

$$\tau = \{(p_i, \rho_i) : i \in I\}$$

for I an index set, and for each $i \in I$, $p_i \in \mathbb{P}$ and ρ_i is a $\mathbb{P}$-name of rank $< \alpha$.

For example, $\emptyset$ is the only $\mathbb{P}$-name of rank 0. So $\{(1. \emptyset)\}$ is a $\mathbb{P}$-name of rank 1.

Definition 6 • *Let $G \subset \mathbb{P}$ and τ a $\mathbb{P}$-name.*

$$\tau_G = \{\, \rho_G : \exists p \in G\,\big((p,\rho) \in \tau\big)\,\}.$$

• $V[G] = \{\, \tau_G : \tau$ *is a* $\mathbb{P}$*-name* $\}$.

The first feature of field extensions we mentioned was **minimality**.

Lemma 1 *If $1 \in G \subset \mathbb{P}$, then $V \subset V[G]$, $G \in V[G]$ and any model of set theory extending V and containing G extends $V[G]$.*

Dem. The last part of the lemma is clear, from the definition of $V[G]$. We define a special class of names to show the first part.

Definition 7 • *Let $x \in V$. The* **canonical** *name for x is*

$$\check{x} = \{\, (1, \check{y}) : y \in x \,\}.$$

• *The canonical name for the generic is*

$$\underline{G} = \{\, (p, \check{p}) : p \in \mathbb{P} \,\}.$$

By induction, $\check{x}_G = x$ for any x, so $\underline{G}_G = G$. □

We would like to know the theory of $V[G]$ (with parameters) inside V itself. Of course, some restrictions are necessary, since $G \notin V$, so in V we cannot know the answer to all the questions of the form '$p \in G$?' for $p \in \mathbb{P}$. So we settle for the second best thing.

Definition 8 *Let $p \in \mathbb{P}$, let $\tau_1, \ldots, \tau_n$ be $\mathbb{P}$-names, and let $\varphi(\vec{\tau})$ be a sentence in the language of forcing. We say that p* **forces** *$\varphi(\vec{\tau})$, $p \Vdash \varphi(\vec{\tau})$, iff*

$$V[G] \models \varphi[\tau_{1G}, \ldots, \tau_{nG}]$$

for all G $\mathbb{P}$-generic over V such that $p \in G$.

For example, if $V \models \varphi[\vec{x}]$ for some Σ_1 statement φ in the language of set theory, then $1 \Vdash \varphi(\check{x}_1, \ldots, \check{x}_n)$.

Notice that the definition of the forcing relation is second order, since it quantifies over all possible $\mathbb{P}$-generic extensions of the universe. So it is not clear that it is definable inside V. In fact, it cannot be, for Tarski's undefinability of truth theorem. Again, we settle for the next best thing:

Theorem 2 *For each n, $\Vdash\restriction_{\Sigma_n}$ is (first-order) definable inside V.*

Remark on the Proof:

We proceed by informal induction on n. For each n, the proof proceeds by induction on the complexity of the formulas in the forcing language. For example, $p \Vdash \neg\varphi$ iff

$$\forall q \leq p\,(q \nVdash \varphi).$$

The most complicated instance of the induction is the atomic case—A feature which basically disappears when doing forcing in recursion theory.

Moreover, 'truth is continuous' among generic extensions:

Theorem 3 *Let G be $\mathbb{P}$-generic over V, let $\tau_1, \ldots, \tau_n$ be $\mathbb{P}$-names, and let $\varphi(\vec{\tau})$ be a sentence in the forcing language. Suppose that*

$$V[G] \models \varphi[\tau_{1G}, \ldots, \tau_{nG}].$$

Then there is $p \in G$ such that $p \Vdash \varphi(\vec{\tau})$.

Of course, this machinery is not very useful if $(V[G], \in)$ is just a partial order.

Theorem 4 $V[G] \models \mathsf{ZFC}$.

Dem. (Sketch) We check some of the axioms.

- $V[G] \models$ **Extensionality**: By definition of τ_G.
- $V[G] \models$ **Foundation**: If $\tau_{1G} \ni \tau_{2G} \ni \ldots$, then the ranks of $\tau_1, \tau_2, \ldots$ form a decreasing sequence of ordinals in V.
- $V[G] \models$ **Pairing**: Given names τ and ρ, let

$$\mu = \{(\mathbf{1}, \tau), (\mathbf{1}, \rho)\}.$$

Then $\mu_G = \{\tau_G, \rho_G\}$.

- $V[G] \models$ **Infinity**: $\omega = \check{\omega}_G \in V[G]$.
- $V[G] \models$ **Union**: Let $x \in V[G]$, say $x = \tau_G$ for some name τ. Let

$$\begin{aligned} \rho = \{ \quad (p, \mu) : \ & \text{there are } q, r \in \mathbb{P} \text{ s.t. } p \leq r, \\ & p \leq q \text{ and for some name } \sigma, \\ & (r, \mu) \in \sigma \text{ and } (q, \sigma) \in \tau \ \}. \end{aligned}$$

Then $\rho_G = \bigcup x$.

- $V \models$ **Comprehension**: Let $x \in V[G]$, say $x = \tau_G$, and let $\varphi(y)$ be a formula in the language of set theory. We need to see that

$$y = \{ z \in x : \varphi(z) \} \in V[G].$$

Let

$$\rho = \{ (p, \mu) \in \mathbb{P} \times \operatorname{dom} \tau : p \Vdash (\mu \in \tau \wedge \varphi(\mu)) \}.$$

Then

$$y = \tau_G. \quad \square$$

III. Models of Set Theory

Now that we have developed this wonderful machinery for adding sets to models of set theory, and still get such models, we need to make sense of it. After all, V being the universe of sets, formally there is no such thing as a generic $G \notin V$.

There are several ways of making sense of the above construction of $V[G]$: Boolean valued models, countable models of fragments of set theory, and a syntactic approach. We expand on the countable models approach and touch briefly on the others:

1. Boolean Valued Models:

Classical models are models with truth values in the trivial Boolean algebra $\{0, 1\}$. Here, we consider models with 'truth values' in a complete Boolean algebra.

We mentioned above that there is an assignment of such algebras to notions of forcing. If $\mathbb{B}(\mathbb{P})$ is the algebra assigned to $\mathbb{P}$, we define a map from formulas to elements of the algebra,

$$\varphi(\vec{\tau}) \mapsto [\![\varphi(\vec{\tau})]\!].$$

Using this map, a $\mathbb{B}(\mathbb{P})$-valued model $V^{\mathbb{B}(\mathbb{P})}$ is defined. It can be shown (from the consistency of our background theory ZFC) that

$$\mathcal{T} = \{\, \varphi(\tau) : [\![\varphi(\vec{\tau})]\!] = 1 \,\}$$

is a (classical) consistent theory extending ZFC and containing the theory of V with parameters. By carefully choosing $\mathbb{P}$, interesting statements can be made part of $\mathcal{T}$. Hence, we can think of $V^{\mathbb{B}(\mathbb{P})}$ as the formal manifestation of $V[G]$.

2. Syntactic Approach:

It is directly verified that $\mathbf{1} \Vdash \varphi$ for each axiom φ of ZFC. It is also verified that given any $p \in \mathbb{P}$,

$$T_p = \{\varphi : p \Vdash \varphi\}$$

is a classical consistent theory and is deductively closed. More carefully, for each n it can be verified that $T_{p,n}$, the restriction of T_p to Σ_n-statements, is consistent. Again, judicious choices of $\mathbb{P}$ lead to interesting statements contained in T_1.

The problem with these approaches is that part of what has to be proven is that the resulting theories are first order, i.e., we must verify the axioms of predicate calculus and its rules of inference together with ZFC and whatever combinatorial statement we are aiming at.

3. Standard Models:

Definition 9 *A model of* ZFC *or a sufficiently large fragment of it is called* **standard** *iff*

1. *it has the form* $\mathcal{M} = (M, \in)$, *i.e., the interpretation of* ϵ *is* $\in\restriction_{M \times M}$, *and*
2. M *is transitive.*

Recall:

Definition 10 *A set* x *is transitive iff every element of* x *is a subset of* x.

It can be justified in a careful way the choice of words 'standard' and why we restrict our attention to such models.

Instead of trying to give such a justification, I will just mention a possible road that heads towards this definition.

The idea is that (by Gödel's completeness theorem—and compactness) if, say, ZFC is consistent, then there are many essentially different models of it. We want to separate among them those which resemble the universe of sets in a closer way (so, by strenghtening the logic, we hope to isolate those statements which must hold in V).

Possible criteria:

We give a list of requirements, each subsuming the previous ones. We say that a class $\mathcal{T}$ of models satisfies a statement iff every model in $\mathcal{T}$ does.

- *Soundness:* $\mathcal{T}$ is Σ^0_1-sound iff any Σ^0_1-statement of arithmetic valid in T is true. We want our models to be Σ^0_1-sound.

Σ^0_1-soundness of a theory is strictly stronger than just consistency. Hilbert's finitism centered around statements of this logical complexity. That's why we want at least such soundness: our models should be *correct* for "real" arithmetic statements.

But it cannot be enough. We want, in particular, to have models which are correct about all arithmetic statements.

- *ω-completeness:* $\mathcal{T}$ is ω-complete iff for every model

  $$\mathcal{A} \in \mathcal{T},$$

  $\mathbb{N}^{\mathcal{A}} \cong \omega$, where $\mathbb{N}^{\mathcal{A}}$ denotes the interpretation of the natural numbers inside $\mathcal{A}$.

  An ω-model is one which is ω-complete. We want to center on ω-models.

Once we have arithmetical correctness, the next step is to look at second order arithmetic.

- *well-foundedness:*

  A β-model is an ω-model $\mathcal{M}$ such that for any Σ^1_1-statement of analysis φ,

$$\mathcal{M} \models \varphi \qquad \text{iff} \qquad \varphi \text{ is true.}$$

Theorem 5 *(Simpson) Given a model $\mathcal{M}$ of set theory,*

$$(\omega, \mathcal{P}(\omega), +, \times, 0, 1, <)^{\mathcal{M}} \text{ is a } \beta\text{-model}$$

iff $(H_{\omega_1}, \in)^{\mathcal{M}}$ is well-founded.

Remark: In fact, Simpson exhibited a fragment of second order arithmetic, ATR_0, such that if $\mathcal{A}$ is a model of ATR_0 then the well-founded trees (in the sense of $\mathcal{A}$) are in correspondence with the elements of a model $\mathcal{M}$ of a weak fragment of set theory + "Every set is Hereditarily Countable".

Our last criterion is a natural strenghtening of the theorem above: We want our models to be well-founded.

Recall:

Definition 11 *A binary relation $<$ is well-founded iff there are no infinite descending chains in $<$, i.e., for no sequence $x_0, x_1, \ldots$ in the field of $<$,*

$$\cdots < x_2 < x_1 < x_0.$$

A structure $\mathcal{M}$ in the language of set theory LST *is well-founded iff $\in^{\mathcal{M}}$ is.*

So we ask of our models to be well-founded. This is good, besides of the correctness reasons suggested above, because well-foundedness gives some kind of canonicity, which leads to our definition of standard:

Theorem 6 *(Mostowski Collapsing Theorem) An* LST*-structure X such that*

$$X \models \mathsf{Extensionality}$$

is well-founded iff it is isomorphic to a transitive set $\mathcal{M}$, i.e.,

$$(X, \in^X) \cong (\mathcal{M}, \in\restriction_{\mathcal{M}\times\mathcal{M}}).$$

Moreover, such isomorphism is unique. In particular

1. *Transitive models of set theory are rigid, i.e., their only automorphism is the identity. In fact, this holds for any transitive set.*

2. *If X is well-founded and $Y \subset X$ is such that $\in^X\restriction_{Y\times Y} = \in\restriction_{Y\times Y}$ and Y is transitive, then the isomorphism is the identity on Y.*

3. *Any* LST*-structure X such that $\in^X = \in\restriction_{X\times X}$ is isomorphic to a transitive set.*

Dem. Given well-founded $X \models$ Extensionality, define inductively a map $\pi : X \to V$ (using Foundation in V) by

$$\pi(x) = \{\, \pi(y) : X \models y \in x \,\},$$

and let $\mathcal{M} = \pi``X$.

Uniqueness follows from considering a minimal counterexample. $\square$

Remark: The theorem holds as well for proper classes X, in the sense that there is a transitive proper class isomorphic to X via the definable isomorphism described above, with the proviso that $\in^X$ is set-like, i.e., that

$$\{\, x \in X : X \models x \in y \,\}$$

is a set for every $y \in X$.

What we need to show now is that we haven't imposed too many restrictions on what we want our models to be.

Of course, if our initial theory is just ZFC, we cannot hope to prove that there are transitive models of it. But for our consistency results that is not necessary: Models of any finite fragment suffice.

Theorem 7 *(Reflexion) Given any finite list $\mathcal{S}$ of sentences that V satisfies, and any ordinal α, there is $\beta > \alpha$ such that*

$$V_\beta \models \mathcal{S}.$$

Dem. We just formalize the Löwenheim-Skolem argument inside ZFC. Let $\mathcal{S}'$ be the closure of $\mathcal{S}$ under subformulas, and take $\beta > \alpha$ such that V_β is closed under Skolem functions for the formulas in $\mathcal{S}'$. $\square$

Remark: If PowerSet $\notin \mathcal{S}$, any H_η, η regular, works as well.

So, suppose we want to prove CH, say, is consistent relative to ZFC. We look for a forcing $\mathbb{P}$ such that in our informal description given above, $V[G] \models \mathsf{CH}$.

For any finite $\mathcal{S} \subset \mathsf{ZFC} + \mathsf{CH}$, the proof that $V[G] \models \mathcal{S}$ only uses finitely many axioms of ZFC. Say, $\mathcal{R} \subset \mathsf{ZFC}$ is enough. Find β such that $V_\beta \models \mathcal{R}$ and $\mathbb{P} \in V_\beta$.

Let $X \prec (V_\beta, \mathbb{P})$ be a countable elementary substructure. Take its Mostowski collapse, and call it $\mathcal{M}$. Let $\overline{\mathbb{P}}$ be the image of $\mathbb{P}$ inside $\mathcal{M}$. Then $\mathcal{P}(\overline{\mathbb{P}})^{\mathcal{M}}$ is countable, so there is a filter G $\overline{\mathbb{P}}$-generic over $\mathcal{M}$. Since

$$\mathcal{M} \models \mathcal{R}$$

and therefore

$$\mathcal{M} \models 1 \Vdash_{\overline{\mathbb{P}}} \mathsf{CH},$$

$\mathcal{M}[G] \models \mathcal{S}$.

Hence, if ZFC is consistent, so is ZFC + CH.

Suggested References

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