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A trichotomy theorem in natural models of AD^+

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Abstract. Assume AD^+ and that either $V = L(\mathcal{P}(\mathbb{R}))$, or $V = L(T, \mathbb{R})$ for some $T \subset \mathsf{ORD}$. Let $(X, \leq)$ be a pre-partially ordered set. Then exactly one of the following cases holds: (1) X can be written as a well-ordered union of pre-chains, or (2) X admits a perfect set of pairwise $\leq$-incomparable elements, and the quotient partial order induced by $(X, \leq)$ embeds into $(2^\alpha, \leq_{lex})$ for some ordinal α, or (3) there is an embedding of $2^\omega/E_0$ into $(X, \leq)$ whose range consists of pairwise $\leq$-incomparable elements.

By considering the case where $\leq$ is the diagonal on X, it follows that for any set X exactly one of the following cases holds: (1) X is well-orderable, or (2) X embeds the reals and is linearly orderable, or (3) $2^\omega/E_0$ embeds into X. In particular, a set is linearly orderable if and only if it embeds into $\mathcal{P}(\alpha)$ for some α. Also, ω is the smallest infinite cardinal, and $\{\omega_1, \mathbb{R}\}$ is a basis for the uncountable cardinals.

Assuming the model has the form $L(T, \mathbb{R})$ for some $T \subset \mathsf{ORD}$, the result is a consequence of $\mathsf{ZF} + \mathsf{DC}_{\mathbb{R}}$ together with the existence of a fine σ-complete measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$ via an analysis of Vopěnka-like forcing. It is known that in the models not covered by this case, $\mathsf{AD}_{\mathbb{R}}$ holds. The result then requires more of the theory of determinacy; in particular, that $V = \mathsf{OD}((<\Theta)^\omega)$, and the existence and uniqueness of supercompactness measures on $\mathcal{P}_{\omega_1}(\gamma)$ for $\gamma < \Theta$.

As an application, we show that (under the same basic assumptions) Scheepers's countable-finite game over a set S is undetermined whenever S is uncountable.

1. Introduction

This paper deals with consequences of the strengthening, AD^+, of the axiom of determinacy, AD, for the general theory of sets, not just for sets of reals or sets of sets of reals.

Particular versions of our results were known either in $L(\mathbb{R})$ or under the additional assumption of $\mathsf{AD}_{\mathbb{R}}$. They can be seen as generalizations of well-known facts in the theory of Borel equivalence relations.

We consider "natural" models of AD^+, namely, those that satisfy $V = L(\mathcal{P}(\mathbb{R}))$, although our results apply to a slightly larger class of models. The special form of V is used in the argument, not just consequences of determinacy.

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Although an acquaintance with determinacy is certainly desirable, we strive to be reasonably self-contained and expect the paper to be accessible to readers with a working understanding of forcing, and combinatorial and descriptive set theory. We state explicitly all additional results we require, and provide enough background to motivate our assumptions. Jech [**12**] and Moschovakis [**18**] are standard sources for notation and definitions. For basic consequences of determinacy, some of which we will use without comment, see Kanamori [**13**].

1.1. Results.

Our main result generalizes the Harrington-Marker-Shelah [**7**] theorem on Borel orderings, the Dilworth decomposition theorem of Foreman [**4**], the Glimm-Effros dichotomy of Harrington-Kechris-Louveau [**6**], and the dichotomy theorem of Hjorth [**10**].

Recall that a *pre-partial ordering* $\leq$ on a set X (also called a *quasi-ordering* on X) is a binary relation that is reflexive and transitive, though not necessarily anti-symmetric. Recall that E_0 is the equivalence relation on 2^ω defined by

$$xE_0y \iff \exists n\,\forall m \geq n\,\big(x(m) = y(m)\big).$$

THEOREM 1.1. *Assume* AD^+ *holds and either* $V = L(T, \mathbb{R})$ *for some* $T \subseteq \mathsf{ORD}$, *or else* $V = L(\mathcal{P}(\mathbb{R}))$. *Let* $(X, \leq)$ *be a pre-partially ordered set. Then exactly one of the following holds:*

(1) X *is a well-ordered union of* $\leq$*-pre-chains.*
(2) There are perfectly many $\leq$*-incomparable elements of* X, *and there is an order preserving injection of the quotient partial order induced by* X *into* $(2^\alpha, \leq_{lex})$ *for some ordinal* α.
(3) There are $2^\omega/E_0$ *many* $\leq$*-incomparable elements of* X.

The argument can be seen in a natural way as proving two dichotomy theorems, Theorems 1.2 and 1.3.

THEOREM 1.2. *Assume* AD^+ *holds and either* $V = L(T, \mathbb{R})$ *for some* $T \subseteq \mathsf{ORD}$, *or else* $V = L(\mathcal{P}(\mathbb{R}))$. *Let* $(X, \leq)$ *be a pre-partially ordered set. Then either:*

(1) There are perfectly many $\leq$*-incomparable elements, or else*
(2) X *is a well-ordered union of* $\leq$*-pre-chains.*

THEOREM 1.3. *Assume* AD^+ *holds and either* $V = L(T, \mathbb{R})$ *for some* $T \subseteq \mathsf{ORD}$, *or else* $V = L(\mathcal{P}(\mathbb{R}))$. *Let* $(X, \leq)$ *be a partially ordered set. Then either:*

(1) There are $2^\omega/E_0$ *many* $\leq$*-incomparable elements of* X, *or else*
(2) There is an order preserving injection of $(X, \leq)$ *into* $(2^\alpha, \leq_{lex})$ *for some ordinal* α.

It is easy to see that $\mathbb{R}$ injects into $2^\omega/E_0$, and it is well-known that, under determinacy, ω_1 does not inject into $\mathbb{R}$, and $2^\omega/E_0$ is not linearly orderable and therefore cannot embed into any linearly orderable set. This shows that the cases displayed above are mutually exclusive.

Theorem 1.2 generalizes a theorem of Foreman [**4**] where, among other results, it is shown (in $\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$) that if $\leq$ is a Suslin/co-Suslin pre-partial ordering of $\mathbb{R}$ without perfectly many incomparable elements, then $\mathbb{R}$ is a union of λ-many Suslin sets, each pre-linearly-ordered by $\leq$, where λ is least such that both $\leq$ and its complement are λ-Suslin.

By considering the case $\leq = \{(x,x) : x \in X\}$, the following corollary, a generalization of the theorem of Silver [**22**] on co-analytic equivalence relations, follows immediately:

THEOREM 1.4. *Assume AD^+ holds and either $V = L(T,\mathbb{R})$ for some $T \subseteq \mathsf{ORD}$, or else $V = L(\mathcal{P}(\mathbb{R}))$. Let X be a set. Then either:*

(1) $\mathbb{R}$ embeds into X, or else
(2) X is well-orderable.

The corollary gives us the following basis result for infinite cardinalities:

COROLLARY 1.5. *Assume AD^+ holds and either $V = L(T,\mathbb{R})$ for some $T \subseteq \mathsf{ORD}$, or else $V = L(\mathcal{P}(\mathbb{R}))$. Let S be an infinite set. Then:*

(1) ω embeds into S.
(2) If κ is a well-ordered cardinal, and S is strictly larger than κ, then either κ^+ or $\kappa \cup \mathbb{R}$ embeds into S. In particular, ω_1 and $\mathbb{R}$ form a basis for the uncountable cardinals. □

Note that there are no assumptions in Theorems 1.2–1.4 on the set X. If, in Theorem 1.4, the set X is a quotient of $\mathbb{R}$ by, say, a projective equivalence relation, one can give additional information on the length of the well-ordering. This has been investigated by several authors including Harrington-Sami [**8**], Ditzen [**2**], Hjorth [**9**], and Schlicht [**21**].

Theorems 1.2 and 1.4 were our original results, and we consider Theorem 1.2 the main theorem of this paper. After writing a first version of the paper, we found Hjorth [**10**], where the version of Theorem 1.4 for $L(\mathbb{R})$ is attributed to Woodin. Hjorth [**10**] investigates in $L(\mathbb{R})$ what happens when alternative 1 in Theorem 1.4 holds but the quotient $\mathbb{R}/E_0$ does not embed into X; much remains to be explored in this area. We remark that the argument of Hjorth [**10**] easily combines with our techniques, so we in fact have Theorem 1.3, a simultaneous generalization of further results in Foreman [**4**], and the main result in Hjorth [**10**]. The following corollary is immediate:

COROLLARY 1.6. *Assume AD^+ holds and either $V = L(T,\mathbb{R})$ for some $T \subseteq \mathsf{ORD}$, or else $V = L(\mathcal{P}(\mathbb{R}))$. Let X be a set. Then either:*

(1) $2^\omega/E_0$ embeds into X, or else
(2) X embeds into $\mathcal{P}(\alpha)$ for some ordinal α. □

In particular:

COROLLARY 1.7. *Assume AD^+ holds and either $V = L(T,\mathbb{R})$ for some $T \subseteq \mathsf{ORD}$, or else $V = L(\mathcal{P}(\mathbb{R}))$. Then a set is linearly orderable if and only if it embeds into $\mathcal{P}(\alpha)$ for some ordinal α.* □

Since it is slightly easier to follow, we arrange the exposition around the proof of Theorem 1.4, and then explain the easy adjustments to the argument that allow us to obtain Theorem 1.2, and the modifications required to the argument in Hjorth [**10**] to obtain Theorem 1.3.

Weak versions of some of these results were known previously in the context of $\mathsf{AD}_\mathbb{R}$. It is thanks to the use of ∞-Borel codes in our arguments that we can extend them in the way presented here.

As an application of our results, we show:

THEOREM 1.8. *Assume* AD^+ *holds and either* $V = L(T, \mathbb{R})$ *for some* $T \subseteq \mathsf{ORD}$, *or else* $V = L(\mathcal{P}(\mathbb{R}))$. *Then the countable-finite game* $CF(S)$ *is undetermined for all uncountable sets* S.

This is a slightly amusing situation in that we have a family of games that are obviously determined under choice, but are undetermined in the natural models of determinacy.

Theorem 1.8 seems of independent interest, since it is still open whether, under choice, player II has a winning 2-tactic in $CF(\mathbb{R})$. Theorem 1.8 seems to indicate that the answer to this question only depends on the cardinal $\mathfrak{c}$ rather than on any particular structural properties of the set of reals.

We also present detailed proofs of two additional results, not due to us. First, directly related to our approach is Woodin's theorem characterizing the ∞-Borel sets:

THEOREM 1.9 (Woodin). *Assume* $\mathsf{ZF} + \mathsf{DC}_{\mathbb{R}} + \mu$ *is a fine* σ*-complete measure on* $\mathcal{P}_{\omega_1}(\mathbb{R})$. *Then a set of reals* A *is* ∞*-Borel iff* $A \in L(S, \mathbb{R})$, *for some* $S \subset \mathsf{ORD}$.

For models of AD^+ of the form $L(T, \mathbb{R})$ for some $T \subset \mathsf{ORD}$, Theorems 1.2 and 1.3 are in fact consequences of the assumptions of Theorem 1.9, this we establish via an analysis of ∞-Borel codes by means of Vopěnka-like forcing.

In the models not covered by this case, $\mathsf{AD}_{\mathbb{R}}$ holds, and the results require two additional consequences of determinacy due to Woodin, namely, that

$$V = \mathsf{OD}((< \Theta)^\omega),$$

and the uniqueness of supercompactness measures on $\mathcal{P}_{\omega_1}(\gamma)$ for $\gamma < \Theta$. We omit the proofs of these two facts.

Second, we also present a proof of the following result of Jackson:

THEOREM 1.10 (Jackson). *Assume* $\mathsf{AC}_\omega(\mathbb{R})$. *Then there is a countable pairing function, i.e, a map*

$$F : [\mathcal{P}(\mathbb{R})]^{\leq\omega} \to \mathcal{P}(\mathbb{R})$$

satisfying:

(1) $F(\mathcal{A})$ *is independent of any particular way* $\mathcal{A}$ *is enumerated, and*
(2) Each $A \in \mathcal{A}$ *is Wadge-reducible to* $F(\mathcal{A})$.

It is because of Theorem 1.10 that our approach to Theorem 1.2 in the $\mathsf{AD}_{\mathbb{R}}$ case is different from the approach when $V = L(T, \mathbb{R})$ for some $T \subseteq \mathsf{ORD}$.

1.2. Organization of the paper.

Section 2 provides the required general background to understand our results, and includes a brief (and perhaps overdue) motivation for AD^+, a quick discussion of the known methods for obtaining natural models of determinacy, and a description of Scheepers's countable-finite game.

In Section 3 we state without proofs some specific consequences of AD^+ that our argument needs. We also prove Jackson's Theorem 1.10.

In Section 4 we prove Woodin's Theorem 1.9, and the dichotomy Theorem 1.4. The argument divides in a natural way into two cases, according to whether $V = L(T, \mathbb{R})$ for some $T \subseteq \mathsf{ORD}$, or $V = L(\mathcal{P}(\mathbb{R}))$. In the latter case, we may also assume $\mathsf{AD}_{\mathbb{R}}$, that we use to derive the result from the former case. The argument in the $\mathsf{AD}_{\mathbb{R}}$ case was suggested by Hugh Woodin. We also explain how to modify the argument to derive our main result, Theorem 1.2, and sketch how to extend the argument in Hjorth [**10**] to prove Theorem 1.3. The deduction of Corollary 1.7 from the argument of Theorem 1.3 is standard.

In Section 5 we analyze the countable-finite game $CF(S)$ in ZF, and use the dichotomy Theorem 1.4 to show that in models of AD^+ of the forms stated above, the game is undetermined for all uncountable sets S. Since trivially player II has a winning strategy if S is countable, this provides us with a complete analysis of the game in natural models of AD^+.

Finally, in Section 6 we close with some open problems.

1.3. Acknowledgments.

We want to thank Marion Scheepers, for introducing us to the countable-finite game, which led us to the results in this paper; Steve Jackson, for allowing us to include in Subsection 3.3 his construction of a pairing function; Matthew Foreman, for making us aware of Foreman [**4**], which led us to improve Theorem 1.4 into Theorem 1.2; and Hugh Woodin, both for developing the beautiful theory of AD^+, and for his key insight regarding the dichotomy Theorem 1.4 in the $\mathsf{AD}_{\mathbb{R}}$ case.

2. Preliminaries

The purpose of this section is to provide preliminary definitions and background. In particular, we present a brief discussion of AD^+ in Subsection 2.2, of two methods for obtaining models of determinacy in Subsection 2.3, and of the countable-finite game in Subsection 2.5.

2.1. Basic notation.

Given a set X, we endow X^ω with the (Tychonoff's) product topology of ω copies of the discrete space X, so basic open sets have the form

$$[s] = \{f \in X^\omega : s \subset f\},$$

where $s \in X^{<\omega}$. This will always be the case, even if X is an ordinal or carries some other natural topology.

$\mathbb{R}$ will always mean Baire space, ω^ω, that is homeomorphic to the set of irrational numbers.

Definition 2.1. *A* ***tree*** *T on a finite product $\prod_{i<n} X_i$ (typically for us, $n = 1$ or 2) is a subset of $(\prod_{i<n} X_i)^{<\omega}$ that is closed under restrictions and such that, whenever $(p_i : i < n) \in T$, then all p_i, $i < n$, have the same length.*

If T is a tree on $X \times Y$ and $x \in X^{<\omega}$, then

$$T_x = \{y \in Y^{<\omega} : (x, y) \in T\}$$

and if $x \in X^\omega$, then

$$T_x = \bigcup_n T_{x\restriction n},$$

so T_x is a tree on Y.

We denote by $[T]$ the set of infinite branches through T and, if T is a tree on $X \times Y$, then

$$p[T] = \{f \in X^\omega : \exists g \in Y^\omega \left((f, g) \in [T]\right)\} = \{f : T_f \text{ is ill-founded}\}.$$

As usual, an *infinite branch* through T is a function $f : \omega \to T$ such that for all n, $f \restriction n \in T$.

2.1.1. *Games.*
We deal with infinite games, all following a similar format: For some (fixed) set X, two players I and II alternate making moves for ω many innings, with I moving first. In each move, the corresponding player plays an element of X :

I	x_0		x_2		$\dots$
II		x_1		x_3	

(Specific games may impose restrictions on what elements are allowed as the play progresses.) This way both players collaborate to produce an element $x = \langle x_0, x_1, x_2, \dots \rangle$ of X^ω.

Given $A \subseteq X^\omega$, we define the game $\Game_X(A)$ by following the format just described, and declaring that player I wins iff $x \in A$.

A *strategy* is a function $\sigma : {}^{<\omega}X \to X$. Player I *follows* the strategy σ iff each move of I is dictated by σ and the previous moves of player II:

I	$\sigma(\langle\rangle)$		$\sigma(\langle x_0\rangle)$		$\sigma(\langle x_0, x_1\rangle)$	
II		x_0		x_1		$\dots$

Similarly one defines when II follows σ. A strategy σ is *winning for I* in a game $\Game$ on X iff, for all $x = \langle x_0, x_1, \dots \rangle \in X^\omega$, player I wins the run

$$\sigma * x$$

of the game, produced by I following σ against player II, who plays x bit by bit. Similarly we define when σ is *winning for II.*

We say that a game is determined when there is a winning strategy for one of the players. When the game is $\Game_X(A)$ for some $A \subseteq X^\omega$, it is customary to say that A *is determined.*

Definition 2.2 (AD). *In* ZF, *the* ***axiom of determinacy,*** AD, *is the statement that all* $A \subseteq \mathbb{R}$ *are determined.*

A standard consequence of AD is the *perfect set property* for sets of reals: Any $A \subseteq \mathbb{R}$ is either countable or contains a perfect subset. It follows that AD is incompatible with the existence of a well-ordering of the reals, and in fact, with the weaker statement $\omega_1 \preceq \mathbb{R}$, that ω_1 injects (or *embeds*) into $\mathbb{R}$.

Since determinacy contradicts the axiom of choice, it should be understood as holding not in the universe V of all sets but rather in particular inner models, such as $L(\mathbb{R})$. When our results below assume, for example, that $V = L(\mathcal{P}(\mathbb{R}))$ and that AD holds, this could then be understood as a result about all inner models M that satisfy $\mathsf{AD} + V = L(\mathcal{P}(\mathbb{R}))$.

2.2. AD^+.
At first the study of models of determinacy might appear to be a strange enterprise. However, as the theory develops, it becomes clear that one is really studying the properties of "definable" sets of reals. The notion of definability is inherently vague; however, under appropriate large cardinal assumptions, any reasonable notion of "A is a definable set of reals" is equivalent to "A is in an inner model of determinacy containing all the reals." Thus the study of properties of definable sets of reals becomes the focus.

2.2.1. *The theory* AD^+.
AD^+ is a strengthening of AD. The theory of models of AD^+ is due to Woodin, see for example Woodin [**24**, Section 9.1]. All unattributed results and definitions in this section are either folklore, or can be safely attributed to Woodin.

The starting point for this study is the collection of Suslin sets.

Definition 2.3. *A set* $A \subseteq X^\omega$ *is* κ-***Suslin*** *iff* $A = p[S]$ *for some tree* S *on* $X \times \kappa$.

A set A is **co-κ-Suslin** if $X^\omega \setminus A$ is κ-Suslin and we say that A is **Suslin/co-Suslin** if A is both κ-Suslin and co-κ-Suslin for some κ. That A is κ-Suslin is also expressed by saying that A *has a* κ*-(semi)-scale*. In this paper, we have no use for scales other than the incumbent Suslin representation, so we say no more about them.

Let

$$S_\lambda = \{A \subseteq \mathbb{R} : A \text{ is } \lambda\text{-Suslin}\}.$$

Being Suslin is obviously one notion of being definable, and the classically studied definable sets of reals are all Suslin assuming enough determinacy or large cardinals. Actually, choice implies that all sets of reals are Suslin, so under choice one actually studies which sets of reals are in S_λ for specific cardinals λ. Without choice, it is not necessarily the case that all sets of reals are Suslin.

Definition 2.4. κ *is a* ***Suslin cardinal*** *iff* $S_\kappa \setminus \bigcup_{\lambda<\kappa} S_\lambda \neq \emptyset$.

For example, one can prove in ZF that the first two Suslin cardinals are ω and ω_1. Also, $S_\omega = \underset{\sim}{\mathbf{\Sigma}}^1_1$, the class of projections of closed sets and, assuming some determinacy, then $S_{\omega_1} = \underset{\sim}{\mathbf{\Sigma}}^1_2$, the class of projections of complements of $\underset{\sim}{\mathbf{\Sigma}}^1_1$ sets.

A classical theorem is that "A is Borel" is equivalent to "A is ω-Suslin/co-Suslin." Being Borel is a notion of definability which is obviously extendible by taking longer well-ordered unions. This leads to the notion of ∞-Borel sets.

The ∞-Borel sets are described carefully below. For now, define "A is ∞-Borel with code (S, ϕ)" to mean that $S \subseteq \mathsf{ORD}$, ϕ is a formula in the language of set theory and, for any $x \in \mathbb{R}$,

$$x \in A \iff L[S, x] \models \phi(S, x).$$

Clearly, if T witnesses that A is Suslin, then T also witnesses that A is ∞-Borel, since

$$x \in A \iff L[T, x] \models T_x \text{ is well-founded}.$$

There are multiple senses in which a code for A is easy to calculate from A, assuming that A is ∞-Borel. One of these will be discussed later and another is given by Theorem 2.5 below.

First, we need a couple of basic notions. Define

$$\Theta = \sup\{|\cdot|_\leq : \ \leq \ \text{is a pre-well-ordering of a subset of } \mathbb{R}\},$$

where $|\cdot|_\leq$ is the rank of the pre-well-ordering $\leq$. Equivalently,

$$\Theta = \sup\{\alpha : \exists f : \mathbb{R} \underset{\text{onto}}{\longrightarrow} \alpha\}.$$

Suppose that A is a set of reals, and define $\underset{\sim}{\mathbf{\Sigma}}^1_1(A)$ as the smallest pointclass containing A and closed under integer quantification, finite unions and intersections, continuous reduction, and existential real quantification. As usual, define $\underset{\sim}{\mathbf{\Pi}}^1_1(A)$ to be the class of complements of $\underset{\sim}{\mathbf{\Sigma}}^1_1(A)$ sets, $\underset{\sim}{\mathbf{\Sigma}}^1_2(A) = \exists^{\mathbb{R}}\underset{\sim}{\mathbf{\Pi}}^1_1(A)$, etc. Each of these classes has a canonical *universal* set $U^1_n(A)$. See Moschovakis [**18**] for the definition of universality, and for this fact.

If $\leq$ is a pre-well-order of length γ, then we say that $S \subseteq \gamma$ is $\underset{\sim}{\mathbf{\Sigma}}^1_n(\leq)$ *in the codes* iff there is a real x such that for $\xi \in \gamma$,

$$\xi \in S \iff \exists y \left[|y|_{\leq} = \xi \text{ and } U^1_n(\leq)(x, y) \right].$$

The Moschovakis Coding Lemma [**18**] states that given any pre-well-order, $\leq$, of $\mathbb{R}$ of length γ, and any $S \subseteq \gamma$, then S is $\underset{\sim}{\mathbf{\Sigma}}^1_1(\leq)$ in the codes.

This yields that if M and N are transitive models of AD with the same reals, and $\gamma < \min\{\Theta^M, \Theta^N\}$, then $\mathcal{P}(\gamma)^M = \mathcal{P}(\gamma)^N$. We then have the following regarding ∞-Borel codes.

Theorem 2.5 (Woodin). *Assume* AD *and that A is ∞-Borel. Then there is a $\gamma < \Theta$, a pre-well-order $\leq$ in $\underset{\sim}{\mathbf{\Pi}}^1_2(A)$ of length γ, and a code $S \subseteq \gamma$ for A. By the coding Lemma, S is $\underset{\sim}{\mathbf{\Sigma}}^1_1(\leq)$ in the codes. So S is $\underset{\sim}{\mathbf{\Sigma}}^1_3(A)$ in the codes, where "$\underset{\sim}{\mathbf{\Sigma}}^1_3(A)$ in the codes" has the obvious meaning.* □

In particular, if M and N are transitive models of AD with $\mathbb{R}^M = \mathbb{R}^N$, and $A \in M$ is Suslin (or just ∞-Borel) in N, then A is ∞-Borel in M, although it need not be the case that A is also Suslin in M.

The following is essentially contained in results of Kechris-Moschovakis-Woodin-Kleinberg [**14**], see also Jackson [**11**].

Theorem 2.6. *Assume* AD*, and suppose that $\lambda < \Theta$ and that A is Suslin/co-Suslin. Then the game $⅁_\lambda(A)$ is determined.* □

Suppose that M is a transitive model of AD, $\lambda < \Theta^M$, and $f : \lambda^\omega \to \omega^\omega$ is in M and continuous. Let A be a set of reals in M, and consider the (A, f)-*induced* game on λ, $⅁_\lambda(f^{-1}[A])$. Suppose moreover that there is a transitive model, N, of AD with the same reals as M, and such that A is Suslin/co-Suslin in N. Then, by Theorem 2.6, in N, $⅁_\lambda(f^{-1}[A])$ is determined and hence, by the Coding Lemma, this game is determined in M, since the winning strategy can be viewed as a subset of λ.

Finally, recall that Suslin subsets $R \subseteq \mathbb{R}^2$ can be *uniformized*, see Moschovakis [**18**], so that there is a partial function $f : \mathbb{R} \to \mathbb{R}$ such that whenever $x \in \mathbb{R}$ and there is a $y \in \mathbb{R}$ with xRy then, in fact, $x \in \text{dom}(f)$ and $xRf(x)$.

Suppose that M is a transitive model of AD, and that $R \subseteq \mathbb{R}^2$ is a relation in M such that for any $x \in \mathbb{R}$ there is a $y \in \mathbb{R}$ such that xRy. If there is a transitive model, N, of AD, with the same reals as M, and such that R is Suslin in N, then R is uniformizable in N. If f is a uniformizing function for R in N, then for any real $x_0 \in N$ there is then a real $x \in N$ coding the sequence $\langle x_n : n < \omega \rangle$ where $x_{n+1} = f(x_n)$ for all $n \in \omega$. Since M and N have the same reals, then x and therefore $\langle x_n : n < \omega \rangle$ are in M. This shows that $\mathsf{DC}_\mathbb{R}$ holds in M.

In summary, we have that if M is a transitive model of AD such that for each $A \in \mathcal{P}(\mathbb{R})^M$, there is a transitive N such that:

(1) N models AD,
(2) N has the same reals as M and,
(3) in N, A is Suslin,

then the following hold in M:

- $\mathsf{DC}_\mathbb{R}$.
- All sets of reals are ∞-Borel.
- For all ordinals $\lambda < \Theta^M$, all continuous functions $f : \lambda^\omega \to \omega^\omega$, and all $A \subseteq \mathbb{R}$, the (A, f)-induced game on λ is determined.

This situation is axiomatized by AD^+.

Definition 2.7 (Woodin). *Over the base theory* ZF, AD^+ *is the conjunction of*

- $\mathsf{DC}_{\mathbb{R}}$.
- *All sets of reals are* ∞*-Borel.*
- $< \Theta$*-ordinal determinacy, i.e., all* (A, f)*-induced games on ordinals* $\lambda < \Theta$ *are determined, for any* $A \subseteq \mathbb{R}$ *and any continuous* $f : \lambda^\omega \to \mathbb{R}$*.*

The following is essentially a consequence of the preceding discussion.

Theorem 2.8. *If* M *is a transitive model of* $\mathsf{ZF} + \mathsf{AD}$ *such that every set of reals in* M *is Suslin in some transitive model* N *of* $\mathsf{ZF} + \mathsf{AD}$ *with the same reals, then* $M \models \mathsf{AD}^+$. □

In fact, in Theorem 2.8, it suffices that M and N satisfy the restriction of ZF to Σ_n sentences, for an appropriate sufficiently large value of n.

Remark 2.9. Suppose that M and N are transitive models of AD with the same reals. Let $\theta = \min\{\Theta^M, \Theta^N\}$. Then, by the Coding Lemma,

$$\Big(\bigcup_{\gamma<\theta} \mathcal{P}(\gamma)\Big)^M = \Big(\bigcup_{\gamma<\theta} \mathcal{P}(\gamma)\Big)^N.$$

In particular, if $A \in M \cap N$ is a set of reals, and A is κ-Suslin in N, for some $\kappa < \Theta^M$, then A is κ-Suslin in M as well.

Recall that *Wadge-reducibility* of sets of reals is given by

$$A \leq_W B$$

iff there is a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $A = f^{-1}[B]$. It is a basic consequence of determinacy that $\leq_W$ is well-founded. We can then assign a rank to each set of reals. The rank of $\leq_W$ itself is exactly Θ. Obviously, a continuous reduction can be coded by a real. With M and N as above, we then have that if $A \in M \cap N$ is a set of reals, then $|A|^M_{\leq_W} = |A|^N_{\leq_W}$. It follows that if A is not Suslin in M but it is Suslin in N, then $\mathcal{P}(\mathbb{R})^M \subset \mathcal{P}(\mathbb{R})^N$ and $\Theta^M < \Theta^N$.

A benefit of considering AD^+ rather than AD is that much of the fine analysis of $L(\mathbb{R})$ under the assumption of determinacy actually lifts to models of the form $L(\mathcal{P}(\mathbb{R}))$ under the assumption of AD^+. Whether AD^+ actually goes beyond AD is a delicate question, still open. We will briefly touch on this below.

2.2.2. $\mathsf{DC}_{\mathbb{R}}$.

Recall that $\mathsf{DC}_{\mathbb{R}}$, or $\mathsf{DC}_\omega(\mathbb{R})$, is the statement that whenever $R \subseteq \mathbb{R}^2$ is such that for any real x there is a y with xRy, then there is a function $f : \omega \to \mathbb{R}$ such that for all n, $f(n)Rf(n+1)$. It is easy to see that this is equivalent to the claim that any tree T on $\mathbb{R}$ with no end nodes has an infinite branch.

Two straightforward (and well-known) observations are worth making: First, in ZF, assume that $\mathsf{DC}_{\mathbb{R}}$ holds and that $T \subseteq \mathsf{ORD}$. Then $\mathsf{DC}_{\mathbb{R}}$ holds in $L(T, \mathbb{R})$. Second, if $\mathsf{DC}_{\mathbb{R}}$ holds in $L(T, \mathbb{R})$ then, in fact, $L(T, \mathbb{R})$ satisfies the axiom of dependent choices, DC.

It is shown in Solovay [**23**] that for models satisfying $V = L(\mathcal{P}(\mathbb{R}))$ and in fact, more generally, for models of $V = \mathsf{OD}(\mathcal{P}(\mathbb{R}))$, if $\mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$ holds, then

$$\mathrm{cf}(\Theta) > \omega \implies \mathsf{DC}.$$

Under AD, there are interesting relationships and variations of $\mathsf{DC}_{\mathbb{R}}$, due to the existence of certain measures. Let $\mathcal{D}$ denote the set of Turing degrees. A set $A \subseteq \mathcal{D}$ is a *cone* iff there is an $a \in \mathcal{D}$ such that

$$A = \{b \in \mathcal{D} : a \leq_T b\},$$

where $\leq_T$ denotes the relation of Turing reducibility. Define the *Martin measure,* μ_M, on $\mathcal{D}$, by

$$A \in \mu_M \iff A \text{ contains a Turing cone.}$$

μ_M is a σ-complete measure on $\mathcal{D}$ and we have

$$\mathsf{DC} \implies \prod \mathsf{ORD}/\mu_M \text{ is well-founded } \implies \prod \omega_1/\mu_M \text{ is well-founded } \implies \mathsf{DC}_{\mathbb{R}}.$$

The first and second implications are trivial. Here is a quick sketch of the third:

Lemma 2.10 (Woodin). *(ZF) Assume that μ_M is a measure, and that $\prod \omega_1/\mu_M$ is well-founded. Then $\mathsf{DC}_{\mathbb{R}}$ holds.*

Proof. Let T be a tree on $\mathbb{R}$. For $d \in \mathcal{D}$, let T_d be the tree T restricted to nodes recursive in d. T_d is in essence a tree on ω and since $\mathsf{DC}_\omega(\omega)$ certainly holds, T_d is ill-founded iff T_d has an infinite branch. If T_d is ill-founded for any d, then there is an infinite branch through T, so assume that all trees T_d are well-founded. For each $\vec{x} \in \mathbb{R}^{<\omega}$, we can define a partial function

$$h_{\vec{x}} : \mathcal{D} \to \omega_1$$

by

$$h_{\vec{x}}(d) = \mathrm{rk}_{T_d}(\vec{x}),$$

leaving $h_{\vec{x}}(d)$ undefined if $\vec{x} \notin T_d$. Note that $h_{\vec{x}}(d)$ *is* defined for μ_M-a.e. degree d.

By assumption, $[h_{\vec{x}}]_{\mu_M}$ is an ordinal $\alpha_{\vec{x}}$, and the map

$$\vec{x} \mapsto \alpha_{\vec{x}}$$

ranks the original tree T and hence T is not a counterexample to $\mathsf{DC}_{\mathbb{R}}$. □

Clearly, in this argument, the Turing degree measure could be replaced by any σ-complete, fine measure, μ, on $\mathcal{P}_{\omega_1}(\mathbb{R})$ satisfying that $\prod \omega_1/\mu$ is well-founded.

Under $\mathsf{AD}^+ - \mathsf{DC}_{\mathbb{R}}$ we actually have the equivalence

$$\prod \mathsf{ORD}/\mu_M \text{ is well-founded } \iff \mathsf{DC}_{\mathbb{R}}.$$

The left hand side was part of Woodin's original formalization of AD^+.

There are models of $\mathsf{AD}^+ + \mathrm{cf}(\Theta) = \omega$. In these models, DC fails, so just the well-foundedness of ultrapowers by fine measures on $\mathcal{P}_{\omega_1}(\mathbb{R})$ does not give DC.

2.2.3. *∞-Borel sets.*

Essentially the ∞-Borel sets are the result of extending the usual Borel hierarchy by allowing arbitrary well-ordered unions.

Work in ZF. Without choice it is better to work with "codes" for sets (descriptions of their transfinite Borel construction) rather than with the sets themselves (the output of such a construction), hence an ***∞-Borel set** is any set with an **∞-Borel code.*** For example, it might be the case that for all $\alpha < \gamma$, A_α is ∞-Borel, but there is no sequence of codes c_α and hence $\bigcup_{\alpha<\gamma} A_\alpha$ might not be ∞-Borel.

There are several equivalent definitions of ∞-Borel codes. For definiteness, we present an official version, and then some variants, and leave it up to the reader to check that the notions are equivalent, and even locally equivalent when required.

DEFINITION 2.11. *Fix a countable set of objects*

$$N = \{\neg, \bigvee\} \cup \{\dot{n} : n \in \omega\}$$

*with N disjoint from ORD; e.g., $\neg = (0,0)$, $\bigvee = (0,1)$, and $\dot{n} = (1,n)$ would suffice. The **∞-Borel codes** (BC) are defined recursively by: $T \in \mathsf{BC}$ iff one of the following holds:*

- $T = \langle \dot{n} \rangle$.
- $T = \bigvee_{\alpha<\kappa} T_\alpha = \{\langle \bigvee, \alpha\rangle^\frown s : s \in T_\alpha\}$ *where each* $T_\alpha \in \mathsf{BC}$.
- $T = \neg S = \{\langle \neg \rangle^\frown s : s \in S\}$ *where* $S \in \mathsf{BC}$.

Clearly the codes form a subset of the well-founded trees on $\kappa \cup N$. Set

$$\mathsf{BC}_\kappa = \mathsf{BC} \cap \{T : T \text{ is a well-founded tree of rank } < \kappa\}.$$

For κ a limit ordinal, BC_κ is closed under finite joins. If $\mathrm{cf}(\kappa) > \omega$, then BC_κ is σ-closed and, if κ is regular, then BC_κ is $< \kappa$-closed. Clearly for regular κ, $\mathsf{BC}_\kappa = \mathsf{BC} \cap H(\kappa)$.

DEFINITION 2.12. *A set of reals is **∞-Borel** iff it is the interpretation of some $T \in \mathsf{BC}$. We denote this interpretation by A_T, and define it by recursion as follows:*

- $A_{\dot{n}} = \{x \in 2^\omega : x(n) = 1\}$.
- $A_{\bigvee_{\alpha<\kappa} T_\alpha} = \bigcup_{\alpha<\kappa} A_{T_\alpha}$.
- $A_{\neg T} = 2^\omega \setminus A_T$.

The predicates "$T \in \mathsf{BC}$" and "$x \in A_T$" are Σ_1 and absolute for any model of $\mathsf{KP} + \Sigma_1$-separation. (Just KP is not enough, since the code must be well-founded.)

Let $\mathbb{B}_\infty$ denote the collection of ∞-Borel sets, and let $\mathbb{B}_\kappa$ be the subset of $\mathbb{B}_\infty$ consisting of those sets with codes in BC_κ. In particular, if ω_1 is regular, then $\mathbb{B}_{\omega_1}$ is just the algebra of Borel sets.

The following gives a few alternate definitions for the ∞-Borel sets. The equivalence of the first three is local in the sense that it is absolute to models of $\mathsf{KP} + \Sigma_1$-separation. The equivalence with the fourth one is still reasonably local, certainly absolute to models of ZF.

- A is ∞-Borel.
- There is a tree T on $\kappa \times 2$ such that $A(x)$ iff player I wins the game $\Game_{T,x}$ given by: Players I and II take turns playing ordinals $\alpha_i < \kappa$ so in the end they play out $f \in \kappa^\omega$. Player I wins iff $(f, x) \in [T]$.

  The game $\Game_{T,x}$ is closed for I and hence determined. (In this case T is taken as the code and $A_T = \{x : \text{I wins } \Game_{T,x}\}$.)
- There is a Σ_1 formula ϕ (in the language of set theory, with two free variables) and $S \subseteq \gamma$ for some γ, such that

  $$A(x) \iff L[S,x] \models \phi(S,x).$$

  (Here (ϕ, S) is taken to be the code and $A = A_{\phi,S}$ is the set coded.)
- There is a formula ϕ and $S \subseteq \gamma$ for some γ, such that

  $$A(x) \iff L[S,x] \models \phi(S,x).$$

  (Once again, (ϕ, S) is taken to be the code and $A = A_{\phi,S}$ is the set coded.)

It is thus natural to identify codes with sets of ordinals, and we will often do so.

For example, as mentioned above, Suslin sets are ∞-Borel. On the other hand, Suslin subsets of $\mathbb{R} \times \mathbb{R}$ can be uniformized, while in general there can be non-uniformizable sets in a model of AD^+, so it is not true that all ∞-Borel sets are Suslin.

Under fairly mild assumptions, being ∞-Borel already entails many of the nice regularity properties shared by the Borel sets. In particular, suppose that S is a code witnessing that A_S is ∞-Borel, and suppose that

$$|\mathcal{P}(\mathbb{P}_c) \cap L[S]|^V = \omega,$$

where $\mathbb{P}_c$ is the Cohen poset (essentially $\omega^{<\omega}$). Then A_S has the property of Baire. Similarly, if $|\mathcal{P}(\mathbb{P}_L) \cap L[S]|^V = \omega$, where $\mathbb{P}_L$ is random forcing, then A_S is Lebesgue measurable. In general, if ω_1^V is inaccessible in $L[S]$, then A_S has all the usual regularity properties.

Note that Theorem 1.9 provides us, over the base theory $\mathsf{ZF} + \mathsf{DC}_{\mathbb{R}}+$"there is a fine measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$," with yet another equivalence for the notion of ∞-Borel; however, we know of no reasonable sense in which this version would be local as the previous ones.

2.2.4. *Ordinal determinacy.*
AD states that all games on ω are determined. One may wonder whether it is consistent with ZF that, more generally, all games on ordinals are determined. This is not the case; in fact, it is well-known that there is an undetermined game on ω_1.

To see this, consider two cases. If AD fails, we are done, and there is in fact an undetermined game on ω. If AD holds, then $\omega_1 \not\preceq \mathbb{R}$. Consider the game where player I begins by playing some $\alpha < \omega_1$, and player II plays bit by bit a real coding $\omega + \alpha$. Since any countable ordinal can be coded by a real, it is clear that player I cannot have a winning strategy. Were this game determined, player II would have a winning strategy σ. But it is straightforward to define from σ an uncountable sequence of reals, and we reach a contradiction.

It follows that some care is needed in the way the payoff of ordinal games is chosen if we want them to be determined, and this is why $<\Theta$-determinacy is stated as above.

Note that ordinal determinacy indeed implies determinacy, so AD^+ strengthens AD. One consequence of ordinal determinacy that we will use is the following:

THEOREM 2.13 (Woodin). *Assume* $\mathsf{ZF} + \mathsf{AD}^+$. *For every Suslin cardinal* κ, *there is a unique normal fine measure* μ_κ *on* $\mathcal{P}_{\omega_1}(\kappa)$. *In particular,* $\mu_\kappa \in \mathsf{OD}$. $\square$

Let κ be a Suslin cardinal. For any $\gamma < \kappa$, define $\mu_\gamma = \pi_{\kappa,\gamma}(\mu_\kappa)$ where

$$\pi_{\kappa,\gamma} : \mathcal{P}_{\omega_1}(\kappa) \to \mathcal{P}_{\omega_1}(\gamma)$$

is defined by $\sigma \mapsto \sigma \cap \gamma$. This gives a canonical sequence of ω_1-*supercompactness* measures on all γ less than the supremum of the Suslin cardinals.

2.2.5. $\mathsf{AD}_{\mathbb{R}}$.
Over ZF, $\mathsf{AD}_{\mathbb{R}}$ is the assertion that for all $A \subseteq \mathbb{R}^\omega$, the game $\Game_{\mathbb{R}}(A)$ is determined.

$\mathsf{DC}_{\mathbb{R}}$ is an obvious consequence of $\mathsf{AD}_{\mathbb{R}}$, and Woodin has shown that $\mathsf{AD}_{\mathbb{R}}$ yields that all sets of reals are ∞-Borel. However, as far as we know, the only proof of $\mathsf{AD}_{\mathbb{R}} \implies \mathsf{AD}^+$ uses an argument of Becker [**1**] for getting scales from uniformization, and Becker's proof uses DC. The minimal model of $\mathsf{AD}_{\mathbb{R}}$ does not satisfy DC, but does satisfy AD^+; this requires a different argument basically analysing the strength of the least place where $\mathsf{AD} + \neg\mathsf{AD}^+$ could hold. Woodin has shown from $\mathsf{AD}_{\mathbb{R}} + \mathsf{AD}^+$ that all sets are Suslin, without appeal to Becker's argument. At the moment, the lack of a proof (not assuming DC) that $\mathsf{AD}_{\mathbb{R}} \implies \mathsf{AD}^+$, and hence that $\mathsf{AD}_{\mathbb{R}} \implies$ all sets are Suslin , seems to be a weakness in the theory. To make results easy to state, from here on $\mathsf{AD}_{\mathbb{R}}$ will mean $\mathsf{AD}_{\mathbb{R}} + \mathsf{AD}^+$.

Let

$$\kappa_\infty = \sup\{\kappa : \kappa \text{ is a Suslin cardinal}\}.$$

Assuming AD,

$$\kappa_\infty = \Theta \iff \text{all sets of reals are Suslin.}$$

THEOREM 2.14 (Steel, Woodin). *(ZF)*

(1) AD implies that the Suslin cardinals are closed below κ_∞.
(2) $\mathsf{AD}_{\mathbb{R}}$ is equivalent to $\mathsf{AD} + \kappa_\infty = \Theta$.
(3) AD^+ is equivalent to $\mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$ together with "the Suslin cardinals are closed below Θ." □

Thus if there is a model of $\mathsf{AD} + \neg\mathsf{AD}^+$, then in this model $\kappa_\infty < \Theta$ and κ_∞ is not a Suslin cardinal. The main open problem in the theory of AD is whether AD does in fact (over ZF) imply AD^+.

2.2.6. $L(\mathbb{R})$.

It is not immediate even that $L(\mathbb{R}) \models \mathsf{AD} \to \mathsf{AD}^+$. This is the content of the following results:

THEOREM 2.15 (Kechris [**15**]). *Assume $V = L(\mathbb{R}) \models \mathsf{ZF}+\mathsf{AD}$. Then $\mathsf{DC}_{\mathbb{R}}$ (and therefore DC) holds.* □

As mentioned previously, in the context of choice, it is automatic that DC holds in $L(\mathbb{R})$, regardless of whether AD does. Woodin has found a new proof of Kechris's result using his celebrated *derived model theorem*, stated in Subsection 2.3.

The basic fine structure for $L(\mathbb{R})$ yields that, working in $L(\mathbb{R})$, if $\Gamma(x)$ is the lightface pointclass consisting of all sets of reals Σ_1-definable from x, then $\Gamma(x) = \Sigma^2_1(x)$, the collection of all sets of reals, A, such that

$$y \in A \iff \exists B \subseteq \mathbb{R}\, \phi(B, x, y)$$

for some Π^1_2 formula ϕ. As usual, $\Pi^2_1(x)$ is the collection of complements of $\Sigma^2_1(x)$ sets, and $\Delta^2_1(x)$ is the collection of sets that are both $\Sigma^2_1(x)$ and $\Pi^2_1(x)$.

Solovay's basis theorem goes further to assert that the witnessing set can in fact be chosen to be $\Delta^2_1(x)$, that is,

$$x \in A \iff \exists B \in \Delta^2_1(x)\, \phi(B, x).$$

A theorem of Martin and Steel [**17**] give that, under AD, $\Sigma_1^{L(\mathbb{R})}$ has the scale property. For us, this means that every set in $\Sigma_1^{L(\mathbb{R})}$ is Suslin. Combining these two results gives that any $\Sigma_1^{L(\mathbb{R})}$ fact about a real x has a Suslin/co-Suslin witness.

Let n be as in the paragraph following Theorem 2.8. The theory ZF_n resulting from only considering those axioms of ZF that are at most Σ_n sentences, is finitely axiomatizable.

Suppose $L(\mathbb{R})$ failed to satisfy AD^+. Then the following $\Sigma_1^{L(\mathbb{R})}$ statement holds:

$$\exists M \left[\mathbb{R} \subseteq M \text{ and } M \models \mathsf{ZF}_n + \neg\mathsf{AD}^+\right].$$

By the basis theorem together with the Martin-Steel result, the witness M can be coded by a Suslin/co-Suslin set. Thus $M \subseteq L(\mathbb{R})$ are two transitive models of $\mathsf{ZF}_n+\mathsf{AD}$ with the same reals, and one can check that each set of reals in M is Suslin in $L(\mathbb{R})$. It follows from Theorem 2.8 that $M \models \mathsf{AD}^+$ and this is a contradiction. This proves:

Corollary 2.16. *$L(\mathbb{R}) \models \mathsf{AD} \to \mathsf{AD}^+$.* □

Two results that hold for $L(\mathbb{R})$ whose appropriate generalizations are relevant to our results are the fact that in $L(\mathbb{R})$ every set is ordinal definable from a real, and the following:

Theorem 2.17 (Woodin). *$L(\mathbb{R}) \models \exists S \subseteq \Theta\,(\mathsf{HOD} = L[S])$.* □

The set S as in Theorem 2.17 is obtained by a version of Vopěnka's forcing due to Woodin that can add $\mathbb{R}$ to $\mathsf{HOD}^{L(\mathbb{R})}$. Variants of this forcing are very useful at different points during the development of the AD^+ theory, the general version being:

Theorem 2.18 (Woodin). *Suppose that AD^+ holds and that $V = L(\mathcal{P}(\mathbb{R}))$. Then there is $S \subseteq \Theta$ such that $\mathsf{HOD} = L[S]$.* □

S can be taken to code the Σ_1-theory of Θ in $L(\mathcal{P}(\mathbb{R}))$. If $V = L(T, \mathbb{R})$ for some set $T \subseteq \mathsf{ORD}$, then S can be obtained by a generalization of the version of Vopěnka's forcing hinted at above. The stronger statement that $\mathcal{P}(\mathbb{R}) \subseteq L(S, \mathbb{R})$ is false in general. For example, it implies that $\mathsf{AD}_{\mathbb{R}}$ *fails*, as claimed in Woodin [**24**, Theorem 9.22].

2.3. Obtaining models of AD^+.

Here we briefly discuss two methods by which (transitive, proper class) models of AD^+ (that contain al the reals) can be obtained; this illustrates that there is a wide class of natural models to which our results apply:

2.3.1. *The derived model theorem.*

The best understood models of AD^+ come from a construction due to Woodin, the *derived model theorem*. In a precise sense, this is our *only* source of natural models of AD^+.

The derived model theorem carries two parts, first obtaining models of determinacy from Woodin cardinals, and second recovering models of choice with Woodin cardinals from models of determinacy:

Theorem 2.19 (Woodin). (ZFC) *Suppose δ is a limit of Woodin cardinals. Let $V(\mathbb{R}^*)$ be a symmetric extension of V for $\mathtt{Coll}(\omega, <\delta)$, so*

$$\mathbb{R}^* = \bigcup_{\alpha<\delta} \mathbb{R}^{V[G\restriction\alpha]}$$

for some $G \subseteq \mathtt{Coll}(\omega, <\delta)$ generic over V. Then:

(1) $\mathbb{R}^* = \mathbb{R}^{V(\mathbb{R}^*)}$; $V(\mathbb{R}^*) \not\models \mathsf{AC}$; and $V(\mathbb{R}^*) \models \mathsf{DC}$ iff δ is regular.

(2) Define

$$\Gamma = \{\, A \subseteq \mathbb{R}^* : A \in V(\mathbb{R}^*) \text{ and } L(A, \mathbb{R}^*) \models \mathsf{AD}^+ \,\}.$$

Then $L(\Gamma, \mathbb{R}^*) \models \mathsf{AD}^+$. □

Notice that

$$L(\Gamma, \mathbb{R}^*) \models V = L(\mathcal{P}(\mathbb{R}))$$

and that, in particular, the theorem implies $\Gamma \neq \emptyset$.

REMARK 2.20. If δ as above is singular, then $\mathbb{R}^* \subsetneq \mathbb{R}^{V[G]}$.

It is the fact that the theorem admits a converse that makes it the optimal result of its kind, in the sense that it captures all the $L(\mathcal{P}(\mathbb{R}))$-models of AD^+:

THEOREM 2.21 (Woodin). ($\mathsf{ZF} + \mathsf{DC}_{\mathbb{R}}$) *Suppose $V = L(\mathcal{P}(\mathbb{R})) + \mathsf{AD}^+$. There exists $\mathbb{P}$ such that if G is $\mathbb{P}$-generic over V, then in $V[G]$:*

(1) There exists an inner model $N \models \mathsf{ZFC}$,
(2) ω_1^V is limit of Woodin cardinals in N,
(3) $N(\mathbb{R}^V)$ is a symmetric extension of N for $\mathtt{Coll}(\omega, <\omega_1^V)$, and
(4) $V = N(\mathbb{R}^V)$. □

REMARK 2.22. N is not an inner model of V. If it were, every real of V would be in a set-generic extension of a (fixed) inner model of V by a forcing of size $< \omega_1^V$. AD prevents this from happening, as it is a standard consequence of determinacy that any subset of ω_1 is constructible from a real.

The point here is that to be a symmetric extension is first order, as the following well-known result of Woodin indicates:

LEMMA 2.23 (Woodin). *Suppose $N \models \mathsf{ZFC}$, let δ be a strong limit cardinal of N, and let $\sigma \subseteq \mathbb{R}$. Then $N(\sigma)$ is a symmetric extension of N for $\mathtt{Coll}(\omega, <\delta)$ iff*

(1) Whenever $x, y \in \sigma$, then $\mathbb{R} \cap N[x, y] \subseteq \sigma$,
(2) Whenever $x \in \sigma$, then x is $\mathbb{P}$-generic over N for some $\mathbb{P} \in N$ such that $|\mathbb{P}|^N < \delta$, and
(3) $\sup_{x\in\sigma} \omega_1^{N[x]} = \delta$. □

Let us again emphasize that all the models obtained using the construction described in the derived model theorem satisfy $V = L(\mathcal{P}(\mathbb{R}))$, and they also satisfy AD^+.

2.3.2. *Homogeneous trees.*
The second method we want to mention is via homogeneously Suslin representations in the presence of large cardinals. We briefly recall the required definitions. The key notion of *homogeneous tree* was isolated from careful examination of Martin's proof of $\utilde{\mathbf{\Pi}}^1_1$-determinacy from a measurable cardinal.

DEFINITION 2.24. *Let $1 \leq n \leq m < \omega$. For X a set and $A \subseteq X^m$, let*

$$A \restriction n := \{\, u \restriction n : u \in A \,\}.$$

*Let κ be a cardinal, and let μ and ν be measures on κ^n and κ^m, respectively. We say that μ and ν are **compatible** iff*

$$\forall A \subseteq \kappa^m \,(A \in \nu \Rightarrow A \restriction n \in \mu)$$

or, equivalently, iff $B \in \mu \Rightarrow \{\, u \in \kappa^m : u \restriction n \in B \,\} \in \nu$.

DEFINITION 2.25. *Let T be a tree on $\omega \times \kappa$. We say that $\langle\, \mu_u : u \in \omega^{<\omega} \,\rangle$ is a **homogeneity system** for T iff*

(1) For each $u \in \omega^{<\omega}$, μ_u is an ω_1-complete ultrafilter on T_u (i.e., $T_u \in \mu_u$),
(2) For each $u \sqsubseteq v \in \omega^{<\omega}$, μ_u and μ_v are compatible, and

(3) For any $x \in \omega^\omega$, if $x \in p[T]$ and $A_i \in \mu_{x\restriction i}$ for all $i < \omega$, then there is $f : \omega \to \kappa$ such that $\forall i\,(f \restriction i \in A_i)$.

*We say that T is a **homogeneous tree** just in case it admits a homogeneity system, and we say it is κ-**homogeneous** iff it admits a homogeneity system*

$$\langle\, \mu_u : u \in \omega^{<\omega} \,\rangle$$

where each μ_u is κ-complete.

Note that if μ is a homogeneity system for T and $x \notin p[T]$ then, setting $A_i = T_{x\restriction i}$, there is no f such that $\forall i\,(f \restriction i \in A_i)$. Thus, item 3 gives a characterization of membership in $p[T]$.

The key fact relating determinacy and the notion of homogeneous trees is the following:

Theorem 2.26 (Martin). *If $A = p[T]$ for some homogeneous tree T, then $\Game_\omega(A)$ is determined.* □

Definition 2.27. *A set $A \subseteq \omega^\omega$ is **homogeneously Suslin** iff there is a homogeneous tree T such that $A = p[T]$.*

*A is κ-**homogeneously Suslin** (or κ-**homogeneous**) iff it is the projection of a κ-homogeneous tree.*

*A is ∞-**homogeneous** iff it is κ-homogeneous for all κ.*

For example, $\underset{\sim}{\mathbf{\Pi}}^1_1$-sets are homogeneously Suslin: For any measurable κ and any $\underset{\sim}{\mathbf{\Pi}}^1_1$-set A, there is a κ-homogeneous tree T on $\omega \times \kappa$ with $A = p[T]$.

All the proofs of determinacy from large cardinals have actually shown that the pointclasses in question are not just determined, but consist of homogeneously Suslin sets. Under large cardinal hypotheses, the ∞-homogeneous sets are closed under nice operations. For example:

Theorem 2.28 (Martin-Steel (1985)). *Let δ be a Woodin cardinal. Suppose that $A \subseteq \omega^\omega \times \omega^\omega$ is δ^+-homogeneous and*

$$B = \exists^{\mathbb{R}} \neg A := \{x : \exists y\,((x, y) \notin A)\}.$$

Then B is κ-homogeneous for all $\kappa < \delta$. □

This allows us to identify, from enough large cardinals, nice *pointclasses* $\Gamma \subset \mathcal{P}(\mathbb{R})$ such that

$$L(\Gamma, \mathbb{R}) \models \mathsf{AD}.$$

In fact, although this is not a straightforward adaptation of the sketch presented for $L(\mathbb{R})$, the arguments establishing that sets in Γ are (sufficiently) homogeneous also allow one to show that $L(\Gamma, \mathbb{R}) \models \mathsf{AD}^+$.

Notice that, once again,

$$L(\Gamma, \mathbb{R}) \models V = L(\mathcal{P}(\mathbb{R})).$$

A posteriori, it follows that these models arise by applying the derived model theorem to a suitable forcing extension of an inner model of V.

2.4. Canonical models of AD^+.

AD^+ is essentially about sets of reals; in particular, if AD^+ holds, then it holds in $L(\mathcal{P}(\mathbb{R}))$. We informally say that models of this form are *natural* and note that, for investigating global consequences of AD^+, these are indeed the natural inner models to concentrate on.

There are however, other *canonical* inner models of AD^+, typically of the form $L(\mathcal{P}(\mathbb{R}))[X]$ for various nice X. Niceness here means that the models satisfy an appropriate version of *condensation.* For example, $L(\mathbb{R})[\mu]$ where μ a fine measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$ which is moreover normal in the sense of Solovay [**23**]; or $L(\mathbb{R})[\mathcal{E}]$ for $\mathcal{E}$ a coherent sequence of extenders. We will not consider these more general structures in this paper.

As explained in the previous subsection, the best known methods of producing models of determinacy actually give us models of $\mathsf{AD}^+ + V = L(\mathcal{P}(\mathbb{R}))$. Of course, not all known models of AD^+ have a nice canonical form, but they are typically obtained from these models, for example, by going to a forcing extension, as in Woodin's example in Kechris [**15**] of a model of $\mathsf{AD}^+ + \neg\mathsf{AC}_\omega$ obtained by forcing over $L(\mathbb{R})$.

Woodin has shown that any model of AD^+ of the form $L(\mathcal{P}(\mathbb{R}))$ either satisfies $V = L(T, \mathbb{R})$ for some set $T \subset \mathsf{ORD}$, or else it is a model of $\mathsf{AD}_\mathbb{R}$; a precise statement will be given in Theorem 3.1 below. This may help explain the hypothesis in the statement of our main result, Theorem 1.2.

2.5. The game $CF(S)$.

Marion Scheepers [**19**] introduced the *countable-finite game* around 1991. It is a perfect information, ω-length, two-player game relative to a set S. We denote it by $CF(S)$.

I	O_0		O_1		$\dots$
II		T_0		T_1	

At move n, Player I plays O_n, a *countable* subset of S, and Player II responds with T_n, a *finite* subset of S.

Player II wins iff $\bigcup_n O_n \subseteq \bigcup_n T_n$.

Obviously, under choice, Player II has a winning strategy. Scheepers [**19, 20**] investigates what happens when the notion of *strategy* is replaced with the more restrictive notion on *k-tactic* for some $k < \omega$: As opposed to strategies, that receive as input the whole sequence of moves made by the opponent, in a k-tactic, only the previous k moves of the opponent are considered.

Tactics being much more restrictive, additional conditions are then imposed on the players:

- Player I must play increasing sets: $O_0 \subset O_1 \subset \dots$
- Player II wins iff $\bigcup_n O_n = \bigcup_n T_n$.

This setting is not completely understood yet.

- Player I does not have a winning strategy, and therefore no winning k-tactic for any k.
- Player II does not have a winning 1-tactic for any infinite S. (Scheepers [**20**])
- Player II has a winning 2-tactic for S if $|S| < \aleph_\omega$. (Koszmider [**16**])
- Under reasonably mild assumptions (namely, that all singular cardinals κ of cofinality ω are strong limit cardinals and carry a very weak square

sequence in the sense of Foreman-Magidor [**5**]), Player II has a winning 2-tactic for any S. (Koszmider [**16**])
- It is still open whether (in ZFC) Player II has a winning 2-tactic for $CF(\aleph_\omega)$ or for $CF(\mathbb{R})$.

In view of the open problems just mentioned, it is natural to consider the countable-finite game in the absence of choice, to help clarify whether AC really plays a role in these problems.

This was our original motivation for showing the dichotomy Theorem 1.4, so that we could deduce Theorem 1.8 explaining that, under AD^+, in natural models of determinacy, the game $CF(S)$ is undetermined for all uncountable sets S.

3. AD^+

We work in ZF for the remainder of the paper. In this section we state without proof some consequences of AD^+ that we require.

3.1. Natural models of AD^+.

To help explain the hypothesis in Theorem 1.2, we recall the following result:

THEOREM 3.1 (Woodin). *Assume $\mathsf{AD}^+ + V = L(\mathcal{P}(\mathbb{R}))$ and suppose that $\kappa_\infty < \Theta$. Let $T \subset \omega \times \kappa_\infty$ be a tree witnessing that κ_∞ is Suslin. Then $V = L(T^*, \mathbb{R})$ where*

$$T^* = \prod_x T/\mu_T$$

for μ_T the T-degree Martin measure. □

This immediately gives us, via Theorem 2.14:

COROLLARY 3.2 (Woodin). *Assume $\mathsf{AD}^+ + V = L(\mathcal{P}(\mathbb{R}))$. Then either V is a model of $\mathsf{AD}_\mathbb{R}$, or else $V = L(T, \mathbb{R})$ for some set of ordinals T.* □

On the other hand, no model of the form $L(T, \mathbb{R})$ for $T \subseteq \mathsf{ORD}$ can be a model of $\mathsf{AD}_\mathbb{R}$.

Ultrapowers by large degree notions, as in the theorem above, will be essential to establish our result in the $L(T, \mathbb{R})$ case. For models of $\mathsf{AD}_\mathbb{R}$, a different argument is required, and the following result is essential to our approach:

THEOREM 3.3 (Woodin). *Assume $\mathsf{AD}_\mathbb{R} + V = L(\mathcal{P}(\mathbb{R}))$. Then*

$$V = \mathsf{OD}((<\Theta)^\omega),$$

where $(<\Theta)^\omega = \bigcup_{\gamma<\Theta} \gamma^\omega$. □

3.2. Closeness of code to set.

There are a couple of ways that ∞-Borel codes are "close" to the set coded. One way is expressed by Theorem 2.5 above. More relevant to us is the following:

THEOREM 3.4 (Woodin). *Assume $\mathsf{AD}^+ + V = L(\mathcal{P}(\mathbb{R}))$. Let $T \subset \mathsf{ORD}$ and let $A \subseteq \mathbb{R}$ be OD_T. Then A has an OD_T ∞-Borel code.* □

Just as an example of how determinacy can be separated from its structural consequences, the preceding theorem essentially is proved by showing:

THEOREM 3.5 (Woodin). *Suppose $V = L(\mathcal{P}(\gamma)) \models \mathsf{ZF} + \mathsf{DC}$ and μ is a fine measure on $\mathcal{P}_{\omega_1}(\mathcal{P}(\gamma))$ in V, then for all $T \subseteq \mathsf{ORD}$ and $A \subseteq \mathbb{R}$, if $A \in \mathsf{OD}_{T,\mu}$, then A is ∞-Borel with ∞-Borel code in $\mathsf{HOD}_{T,\mu}$.* □

FACT 3.6. *Under* AD *there is an* OD *measure on* $\mathcal{P}_{\omega_1}(\mathcal{P}(\gamma))$ *for all* $\gamma < \Theta$. □

As a corollary, if AD^+ holds, $V = L(\mathcal{P}(\gamma))$ for $\gamma < \Theta$, and $A \in \mathsf{OD}_T \cap \mathcal{P}(\mathbb{R})$, then A has OD_T ∞-Borel code.

This *almost* gives Theorem 3.4 since, assuming $\mathsf{AD}^+ + V = L(\mathcal{P}(\mathbb{R}))$, we have $V = L(\bigcup_{\gamma<\Theta} \mathcal{P}(\gamma))$.

On the other hand, note that Theorem 3.4 is not immediate from Theorem 1.9, even if $V = L(S, \mathbb{R})$.

3.3. A countable pairing function on the Wadge degrees. Our original approach to the dichotomy theorem required the additional assumption that $\mathrm{cf}(\Theta) > \omega$. Both when trying to generalize this approach to the case $\mathrm{cf}(\Theta) = \omega$, and while establishing Theorem 1.8 on the countable-finite game in general, an issue we had to face was whether countable choice for finite sets of reals could fail in a model of $\mathsf{AD}_{\mathbb{R}}$.

That this is not the case follows from the existence of a *pairing function.* Steve Jackson found (in ZF) an example of such a function. Although this is no longer relevant to our argument, we believe the result is interesting in its own right. Below is Jackson's construction.

THEOREM 3.7 (Jackson). *(ZF) There is a function*

$$F : \mathcal{P}(\mathbb{R}) \times \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$$

satisfying:

(1) $F(A, B) = F(B, A)$ for all pairs (A, B), and
(2) Both A and B Wadge-reduce to $F(A, B)$.

PROOF. If $A = B$, simply set $F(A, B) = A$. If $A \subseteq B$ or $B \subseteq A$, set $F(A, B) = (0 * S) \cup (1 * T)$ where S is the smaller of A, B, and T is the larger. Here, $0 * S = \{0^\frown a : a \in S\}$ and similarly for $1 * T$.

If $A \setminus B$ and $B \setminus A$ are both non-empty, we proceed as follows:

Let $X(A, B) \subseteq \mathbb{R}^{\mathbb{Z}}$ be defined by saying that, if $f : \mathbb{Z} \to \mathbb{R}$, then $f \in X(A, B)$ iff there is an i such that $f(i) \in A \setminus B$ (or $B \setminus A$), and for each j, $f(j) \in A$ if $|j - i|$ is even, and $f(j) \in B$ if $|j - i|$ is odd (and reverse the roles of A, B here if $f(i) \in B \setminus A$).

The set $X(A, B)$ is an invariant set (with respect to the shift action of $\mathbb{Z}$ on $\mathbb{R}^{\mathbb{Z}}$), and $X(A, B) = X(B, A)$. (Thus the points of $A \setminus B$ and $B \setminus A$ have to occur at places of different parity; while points of $A \cap B$ can occur anywhere.)

Given $X(A, B)$, we can compute A (and also B) as follows: Fix $z \in A \setminus B$. Then $x \in A$ iff

$$\exists f \in X(A, B)\, \exists i\, \exists j\, (f(i) = z \text{ and } f(j) = x \text{ and } |j - i| \text{ is even}).$$

This shows that A is $\Sigma^1_1(X(A, B))$. If we replace $X(A, B)$ with $X'(A, B)$, the Σ^1_1-jump of $X(A, B)$, then A is Wadge reducible to $X(A, B)$. Finally, we use that there is a Borel bijection between $\mathbb{R}^{\mathbb{Z}}$ and $\mathbb{R}$, and define $F(A, B)$ as the image of $X'(A, B)$ under this map. □

As pointed out by Jackson, essentially the same argument shows the following; recall that $\mathsf{AC}_\omega(\mathbb{R})$ is a straightforward consequence of determinacy, so Theorem 3.8 applies in models of AD:

THEOREM 3.8 (Jackson). *($\mathsf{AC}_\omega(\mathbb{R})$) There is a countable pairing function.*

Proof. Let $\langle A_i : i \in \omega\rangle$ be a sequence of distinct sets of reals. Let $\langle x_i : i \in \omega\rangle$ be a sequence so that for each i there is j such that $x_j \in A_i$, and for each $i \neq j$, there is k such that $x_k \in A_i \triangle A_j$. This can be done with $\mathsf{AC}_\omega(\mathbb{R})$.

For $\langle a_i : i \in \omega\rangle \in \mathbb{R}^\omega$ call $f \in (\mathbb{R} \times 2)^\omega$ ***n*-honest** iff whenever, $f(i) = (x, k)$, then

$$k = 1 \iff x \in A_n,$$

so f is n-honest iff it is a countable approximation to the characteristic function for A_n. Let

$$B = \{f : \exists n\, (f \text{ is } n\text{-honest})\}.$$

Clearly, B does not depend on the ordering of the A_i's.

Let $g_n(2k) = (x_k, 1)$ if $x_k \in A_n$ and $g_n(2k) = (x_k, 0)$ otherwise. Then:

- g_n is the even part of an n-honest function,
- g_n cannot be the even part of a j-honest function for $j \neq n$,
- $x \in A_n \Leftrightarrow \exists f \in B\, \big(f \supset g_n \text{ and } \exists k\, (g_n(k) = (x, 1))\big)$.

This shows that A_n is $\mathbf{\Sigma}^1_1$ in B. □

As a consequence, it follows that for no $\lambda < \Theta$ there is a sequence

$$\langle A_\gamma : \gamma < \lambda\rangle$$

such that each A_γ is a countable subset of $\mathcal{P}(\mathbb{R})$ and $\bigcup_{\gamma<\lambda} A_\gamma$ is cofinal in the Wadge degrees. This is trivial when Θ is regular, but does not seem to be when Θ is singular. Essentially because of this obstacle is that the argument for Theorem 1.2 in the $\mathsf{AD}_\mathbb{R}$ case is different from the argument in the $V = L(T, \mathbb{R})$ case.

4. The dichotomy theorem

Our goal is to establish the dichotomy Theorem 1.4. Our argument utilizes ideas originally due to H. Woodin.

Before we begin, a few words are in order about the way the result came to be. We first proved the dichotomy for models where $V = L(T, \mathbb{R})$ for $T \subseteq \mathsf{ORD}$, and for models of $\mathsf{AD}_\mathbb{R}$ of the form $L(\mathcal{P}(\mathbb{R}))$ where $\mathrm{cf}(\Theta) > \omega$. For the general case, we only succeeded in showing the undeterminacy of the games $CF(S)$. The main additional tool in the $\mathsf{AD}_\mathbb{R}$ case was Theorem 3.3. A key suggestion of Woodin allowed the argument for the dichotomy to be extended to this case as well. The new idea was the weaving together of different well-orderings using the uniqueness of the supercompactness measures for $\mathcal{P}_{\omega_1}(\gamma)$ as γ varies below Θ.

4.1. The $L(T, \mathbb{R})$ case.

We work throughout under the base theory

(BT) $\qquad \mathsf{ZF} + \mathsf{DC}_\mathbb{R} + \mu$ is a fine σ-complete measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$.

It follows from $\mathsf{DC}_\mathbb{R}$ that $L(T, \mathbb{R}) \models \mathsf{DC}$ for all $T \subset \mathsf{ORD}$. So, when working inside models of the form $L(T, \mathbb{R})$, we may freely use DC. In particular, ultrapowers of well-founded models are well-founded. Below, whenever we refer to $L(T, \mathbb{R})$, HOD_S, etc., we will tacitly assume that $T, S \subset \mathsf{ORD}$.

For any $X \in L(T, \mathbb{R})$, there is an $r \in \mathbb{R}$ such that $X \in \mathsf{OD}^{L(T,\mathbb{R})}_{T,r}$. For $\alpha \in \mathsf{ORD}$, let X_α consist of those elements of X definable in $L(T, \mathbb{R})$ from T, r, α, and some real t. If $|\mathbb{R}| \leq |X_\alpha|$, then we are done, so suppose $|\mathbb{R}| \not\leq |X_\alpha|$ for all α.

Define a map from $\mathbb{R}$ onto $X_\alpha \cup \{\emptyset\}$ by setting $x_{t,\alpha}$ to be the $\mathsf{OD}_{T,r}^{L(T,\mathbb{R})}$-least element of X definable from T, r, α, and t, if such an element exists, and otherwise $x_{t,\alpha} = \emptyset$. Let

$$t\, E_\alpha\, t' \iff x_{t,\alpha} = x_{t',\alpha},$$

so E_α is an $\mathsf{OD}_{T,r}^{L(T,\mathbb{R})}$ equivalence relation on $\mathbb{R}$. Clearly the map

$$\phi_\alpha : \mathbb{R}/E_\alpha \xrightarrow[\text{onto}]{\text{1-1}} X_\alpha \cup \{\emptyset\}$$

is $\mathsf{OD}_{T,r}^{L(T,\mathbb{R})}$, thus if we show that $\mathbb{R}/E_\alpha \subset \mathsf{OD}_{T,r,\mu}$, then it follows that

$$X_\alpha \subset \mathsf{OD}_{T,r,\mu}.$$

Consequently, $X \subset \mathsf{OD}_{T,r,\mu}$, and so clearly X is well-orderable.

Definition 4.1. *An equivalence relation E on $\mathbb{R}$ in* ***thin*** *iff $\mathbb{R} \not\preceq \mathbb{R}/E$. Otherwise, E is* ***thick****.*

The theorem we prove is

Theorem 4.2. *Suppose BT and E is an $\mathsf{OD}_{T,r}^{L(T,\mathbb{R})}$ thin equivalence relation, then $\mathbb{R}/E \subset \mathsf{OD}_{T,r,\mu}$.*

4.1.1. *The extent of ∞-Borel sets.*
The proof goes through an analysis of ∞-Borel sets.

Here we show that, assuming BT, every $A \subset \mathbb{R}$ in $L(S,\mathbb{R})$ is ∞-Borel. To show this, it suffices to show that the ∞-Borel sets are closed under $\exists^{\mathbb{R}}$. Once this has been established, the result follows by induction over the levels $L_\alpha(S,\mathbb{R})$ and, for each such level, by induction in the complexity of the definitions of new sets of reals.

Remark 4.3. It is clear that, in $L(S,\mathbb{R})$, every set comes with a description of how to build that set using well-ordered unions, negations, and the quantifier "$\exists^{\mathbb{R}}$".

That every $A \subset \mathbb{R}$ in $L(S,\mathbb{R})$ actually admits an $\mathsf{OD}_{S,\mu}$ ∞-Borel code requires an additional argument, since it is not clear that ∞-Borel sets are closed under well-ordered unions, due to an inability to uniformly pick codes. We omit this additional argument since it would take us too far from our intended goal.

There are in general many descriptions attached to a single set, but the point is that to each description for a set of reals we can attach an ∞-Borel code so long as we have a way to pass from an ∞-Borel code of A_S to one for $\exists^{\mathbb{R}} A_S$.

Notice that we are not claiming that $L(S,\mathbb{R})$ thinks that every set is ∞-Borel; in particular, $\exists^{\mathbb{R}} S$ (see below) might not be in $L(S,\mathbb{R})$. One would need μ to be in $L(S,\mathbb{R})$ to get that all sets in $L(S,\mathbb{R})$ have ∞-Borel code in $L(S,\mathbb{R})$. This is the case under AD where μ is just the Martin measure.

We will in fact show how to pass from an ∞-Borel code S for A to an ∞-Borel code for $\exists^{\mathbb{R}} A_S$ which we will call $\exists^{\mathbb{R}} S$. The map $S \mapsto \exists^{\mathbb{R}} S$ is $\mathsf{OD}_{S,\mu}$.

If $\mu_S = \pi_S(\mu)$ where $\pi_S : \mathcal{P}_{\omega_1}(\mathbb{R}) \to \mathcal{P}_{\omega_1}(\mathbb{R})$ is defined by

$$\pi_S(\sigma) = \mathbb{R} \cap L(S,\sigma),$$

then μ_S is a fine measure. That $\mathbb{R} \cap L(S,\sigma)$ is countable follows from the following discussion, since $\sigma \in L[S,x]$ for some real x.

Let $\kappa = \omega_1^V$, and note that κ is measurable in V, since $\pi : \mathcal{P}_{\omega_1}(\mathbb{R}) \to \omega_1$ given by

$$\sigma \mapsto \sup_{x\in\sigma} \omega_1^{\mathrm{ck}}(x)$$

gives a σ-complete (hence κ-complete) measure $\nu = \pi(\mu)$ on κ. It is clear that ν is non-principal, so κ is indeed measurable.

The fact that κ is measurable in V yields that κ is (strongly) Mahlo in every inner model of choice. Now let N be any set of ordinals, that one can think of as a well-ordered inner model of choice with a fixed well-ordering of the entire structure. Clearly,

$$\mathsf{HOD}_{N,\nu} \subseteq \mathsf{HOD}_{N,\mu},$$

and this model thinks that κ is measurable. Since $N \subset \mathsf{HOD}_{N,\mu}$ is an inner model of $\mathsf{HOD}_{N,\mu}$, $N \models \kappa$ is Mahlo. The claim holds since, if

$$S = \{\gamma < \kappa : \gamma \text{ is } N\text{-inaccessible}\}$$

is N-non-stationary, then $S \notin \nu$.

For any $\sigma \in \mathcal{P}_{\omega_1}(\mathbb{R})$ let

$$H_S^\sigma = \mathsf{HOD}_S^{L(S,\sigma)}$$

and let κ_S^σ be the least inaccessible δ of H_S^σ such that $\delta \geq \Theta^{L(S,\sigma)}$. Define $\sim_S^\sigma$ on the ∞-Borel codes of H_S^σ as follows: For $T, T' \in \mathsf{BC}_S^\sigma$, set

$$T \sim_S^\sigma T' \iff (A_T = A_{T'})^{L(S,\sigma)}.$$

Let

$$\mathbb{Q}_S^\sigma = \mathsf{BC}_S^\sigma / \sim_S^\sigma .$$

$\mathbb{Q}_S^\sigma$ is like the Vopěnka algebra of $L(S,\sigma)$, except that OD ∞-Borel sets are used in place of OD subsets of $\mathbb{R}$. This is made clear by the following lemma whose easy proof we leave to the reader:

Lemma 4.4. *For $x \in L(S,\sigma)$ let*

$$G_S^\sigma(x) = \{b \in \mathbb{Q}_S^\sigma : x \in A_b^{L(S,\sigma)}\}.$$

Then $G_S^\sigma(x)$ is H_S^σ-generic and

$$H_S^\sigma[x] = H_S^\sigma[G_S^\sigma(x)].$$

Moreover, for any $b \in \mathbb{Q}_S^\sigma$ with $b \neq 0_{\mathbb{Q}_S^\sigma}$ there is $x \in L(S,\sigma)$ with $b \in G_S^\sigma(x)$. □

In H_S^σ, set

$$\bar{\mathbb{Q}}_S^\sigma = \mathsf{BC}_{\kappa_S^\sigma,S}^\sigma / \sim_S^\sigma .$$

$\bar{\mathbb{Q}}_S^\sigma$ is $\Theta^{L(S,\sigma)}$-cc since otherwise there would be a sequence $\langle b_\alpha : \alpha < \kappa_S^\sigma\rangle$ of non-zero and incompatible elements in $\bar{\mathbb{Q}}_S^\sigma$. But then, in $L(S,\sigma)$, $\langle A_{b_\alpha} : \alpha < \kappa_S^\sigma\rangle$ would give a pre-well-order of $\mathbb{R}^{L(S,\sigma)}$ of length $\geq \Theta^{L(S,\sigma)}$.

Since κ_S^σ is regular, $\bar{\mathbb{Q}}_S^\sigma$ is κ_S^σ-cc and κ_S^σ-complete, and therefore $\bar{\mathbb{Q}}_S^\sigma$ is complete. So $\bar{\mathbb{Q}}_S^\sigma = \mathbb{Q}_S^\sigma$ and we may identify $\mathbb{Q}_S^\sigma$ with a subset of κ_S^σ in H_S^σ.

Moreover, κ_S^σ is inaccessible and $\mathbb{Q}_S^\sigma$ is κ_S^σ-cc, so we have a canonical enumeration

$$\mathcal{D}_S^\sigma = \langle D_{S,\alpha}^\sigma : \alpha < \kappa_S^\sigma\rangle$$

of maximum antichains of $\mathbb{Q}_S^\sigma$ in H_S^σ. In fact, we enumerate every sequence

$$\langle T_\gamma : \gamma < \alpha\rangle$$

from $\mathsf{BC}_{\kappa_S^\sigma,S}^\sigma$ that becomes such an antichain upon moding out by $\sim_S^\sigma$.

Again, $\mathcal{D}^\sigma_S$ can be coded in a canonical way by a subset of κ^σ_S in H^σ_S. Let b^σ_S be the "minimal" element of $\mathsf{BC}^\sigma_{\kappa^\sigma_S,S}$ such that $b^\sigma_S \sim^\sigma_S S$, and define S^σ as

$$\bigwedge_{\alpha<\kappa^\sigma_S}\ \bigwedge_{T,T'\in D^\sigma_{S,\alpha}} \neg(T\wedge T')\wedge b^\sigma_S \wedge \bigwedge_{\alpha<\kappa^\sigma_S}\bigvee D^\sigma_{S,\alpha}.$$

Modulo $\sim^\sigma_S$, S^σ is just b^σ_S, but before passing to the quotient, we have:

LEMMA 4.5. *For any real x (anywhere)*

$$x\in A_{S^\sigma} \iff x \text{ is } H^\sigma_S \text{ generic over } \mathbb{Q}^\sigma_S \text{ and } H^\sigma_S[x]\models x\in A_S.$$

PROOF. Suppose $x\in A_{S^\sigma}$, and define

$$G^\sigma_S(x)=\{b\in Q^\sigma_S : \exists\alpha\,\exists T\in D^\sigma_{S,\alpha}(x\in T \text{ and } b\sim^\sigma_S T)\}.$$

Clearly this $G^\sigma_S(x)$ meets every antichain of $\mathbb{Q}^\sigma_S$ in H^σ_S. If $T,T'\in G^\sigma_S(x)$, then T,T' are compatible in $\mathbb{Q}^\sigma_S$, since otherwise there is $D^\sigma_{S,\alpha}$ with T,T' in $D^\sigma_{S,\alpha}$, but S^σ explicitly says that x is not in two distinct elements of D^σ_S. So $G^\sigma_S(x)$ is H^σ_S-generic.

$(H^\sigma_S)^{\mathbb{Q}^\sigma_S} \models$ "$\dot x\in A_{b^\sigma_S} \iff \dot x\in A_S$" since this holds for all $x\in L(S,\sigma)$. It follows that

$$H^\sigma_S[x]\models \text{“}x\in A_{b^\sigma_S} \iff x\in A_S\text{”}$$

and, by choice of S^σ, $H^\sigma_S[x]\models x\in A_{b^\sigma_S}$ and thus $H^\sigma_S[x]\models x\in A_S$. This finishes the left to right direction, the converse is easier. □

So whereas $A_{b^\sigma_S}$ only needs to agree with A_S on reals of $L(S,\sigma)$, S^σ has very strong agreement with A_S even on reals in outer models of V.

We are now in a position to establish Woodin's Theorem 1.9, that assuming BT, A is ∞-Borel iff $A\in L(S,\mathbb{R})$, for some $S\subset\mathsf{ORD}$. This follows imediately from the following:

LEMMA 4.6. *Assume BT and let $S\subset\mathsf{ORD}$ be an ∞-Borel code, and view this code as a code for a subset of $\mathbb{R}^2$. Then there is a canonical ∞-Borel code $\exists^{\mathbb{R}}S$ such that*

$$\exists y\,(x,y)\in A_S \iff x\in A_{\exists^{\mathbb{R}}S}.$$

PROOF. The point is that

$$\exists y\,(x,y)\in A_S \iff \text{for } \mu\text{-a.e. } \sigma,\ \big(H^\sigma_S[x]\big)^{\mathtt{Coll}(\omega,\kappa^\sigma_S)}\models \exists y\, A_{S^\sigma}(x,y).$$

In the right to left direction, fix a g in V generic for $\mathtt{Coll}(\omega,\kappa^\sigma_S)$ such that

$$H^\sigma_S[x][g]\models \exists y\,(x,y)\in A_{S^\sigma}.$$

Since $(x,y)\in A_{S^\sigma}$, then $H^\sigma_S[x,y]\models (x,y)\in A_S$ by the previous lemma. So $(x,y)\in A_S$ and hence $\exists y\,(x,y)\in A_S$.

For left to right, just fix y so that $A_S(x,y)$, and take any σ with $x,y\in\sigma$. Then (x,y) is generic over H^σ_S for $\mathbb{Q}^\sigma_S$ and hence satisfies S^σ. It is a $\Sigma^1_1(x,b)$ statement about any real coding S^σ that there is a real y such that $(x,y)\in A_{S^\sigma}$. Thus there is such a real in $H^\sigma_S[x][g]$ for any g enumerating S^σ.

It should be noted that we do not need to use all of H^σ_S above. Instead, we could work with $L[S^\sigma]$, that is

$$\exists y(x,y)\in A_S \iff \forall^*_\mu\sigma\, L[S^\sigma,x]^{\mathtt{Coll}(\omega,\kappa^\sigma_S)}\models \exists y\,(x,y)\in A_{S^\sigma}.$$

Set

$$L[S^\infty, x] = \prod_\sigma L[S^\sigma, x]/\mu.$$

Then

$$\exists y\,(x,y) \in A_S \iff L[S^\infty, x] \models \exists y\,(x,y) \in A_{S^\infty} \iff L[S^\infty, x] \models \varphi(S^\infty, x),$$

so S^∞ together with φ "is" the ∞-Borel code $\exists^{\mathbb{R}} S$. $\square$

Notice that what we actually showed here was that from a description d_A of how to build a set of reals in $L(S,\mathbb{R})$ we can canonically pass to an ∞-Borel code S_{d_A} associated to that description. A and d_A are in $L(S,\mathbb{R})$, and in fact $\mathsf{OD}_{S,t}^{L(S,\mathbb{R})}$, while S_{d_A} is in V, and in fact $\mathsf{OD}_{S,t,\mu}$. This clearly generalizes so that given a sequence of sets of reals $\vec{A} = \langle A_\alpha : \alpha < \gamma\rangle \in L(S,\mathbb{R})$ and an associated description, $d_{\vec{A}} \in \mathsf{OD}_{S,t}^{L(S,\mathbb{R})}$, we produce a corresponding sequence, $\vec{S}$, of ∞-Borel codes, with $\vec{S} \in \mathsf{OD}_{S,t,\mu}$.

Remark 4.7. This argument should illustrate the general technique behind our approach and, really, behind many applications of determinacy that rely on ∞-Borel sets. Namely, the "localization" of ∞-Borel sets we established allows one to argue about them as if they were actually Borel sets, and then lift the results via absoluteness. The proofs of Theorems 1.2–1.4 are further illustrations of this idea.

4.1.2. *The first dichotomy.*

Theorem 4.8. *Suppose* BT. *Then, for every* $X \in L(T,\mathbb{R})$, *if* $|\mathbb{R}| \not\leq |X|$, *then* $X \subset \mathsf{OD}_{T,t,\mu}$ *for some* $t \in \mathbb{R}$.

For $X \in L(T,\mathbb{R})$, X is $\mathsf{OD}_{T,t}^{L(T,\mathbb{R})}$ for some $t \in \mathbb{R}$ and the conclusion of the preceding could be strengthened to $X \in \mathsf{OD}_{T,t,\mu}$ for any $t \in \mathbb{R}$ such that $X \in \mathsf{OD}_{T,t}^{L(T,\mathbb{R})}$.

First we make a useful reduction to equivalence relations on reals. For $X \in \mathsf{OD}_{T,t}^{L(T,\mathbb{R})}$ and $\alpha \in \mathsf{ORD}$, let X_α be the collection of elements of X definable in $L(T,\mathbb{R})$ from α and a real. Take γ so that $X = \bigcup_{\alpha<\gamma} X_\alpha$. To each X_α we can canonically associate an equivalence relation E_α on $\mathbb{R}$ and a bijection $\phi_\alpha : \mathbb{R}/E_\alpha \xrightarrow[\text{onto}]{\text{1-1}} X_\alpha$ with $\phi_\alpha, E_\alpha \in \mathsf{OD}_{T,t}^{L(T,\mathbb{R})}$. We have that $\langle E_\alpha : \alpha < \gamma\rangle$ is an $\mathsf{OD}_{T,t}^{L(T,\mathbb{R})}$-sequence of sets of reals and so, by the comment at the end of the preceding subsection, we get a sequence $\vec{S} = \langle S_\alpha : \alpha < \gamma\rangle$ of ∞-Borel codes with $\vec{S} \in \mathsf{OD}_{T,t,\mu}$.

Theorem 4.9. *Suppose* BT. *If* E *is a thin* ∞*-Borel equivalence relation with code* S, *then* $\mathbb{R}/E \subset \mathsf{OD}_{S,\mu}$.

This will complete the argument: If $|\mathbb{R}| \not\leq X_\alpha$ for all α, then $\mathbb{R}/E_\alpha \subset \mathsf{OD}_{S_\alpha,\mu} \subset \mathsf{OD}_{T,t,\mu}$. So $X_\alpha \subset \mathsf{OD}_{T,t,\mu}$ for all $\alpha < \gamma$ and hence $X \subset \mathsf{OD}_{T,t,\mu}$ as claimed.

Proof. Fix S an ∞-Borel code of a thin equivalence relation. We will use the previously established notation, so $H_S^\sigma = \mathsf{HOD}_S^{L(S,\sigma)}$, $\mathbb{Q}_S^\sigma$, etc. Let $\mathbb{Q}_S^\infty$ be the ultrapower of $\mathbb{Q}_S^\sigma$. It is clear, using Łoś' theorem, that the following hold:

- Every real in V is $\mathbb{Q}_S^\infty$-generic over H_S^∞, since

$$\forall^*\sigma\,(x \text{ is } \mathbb{Q}_S^\sigma\text{-generic over } H_S^\sigma).$$

- Similarly, for $T, T' \in \mathbb{Q}_S^\infty$,
$$T \sim_S^\infty T' \iff (A_T = A_{T'})^V,$$
so $\mathbb{Q}_S^\infty$ is a subalgebra of $\mathbb{B}_\infty$.

Define
$$W_S^\infty = \{(p,p) \in \mathbb{Q}_S^\infty \times \mathbb{Q}_S^\infty : \Vdash^{H_S^\infty}_{\mathbb{Q}_S^\infty \times \mathbb{Q}_S^\infty} \dot{r}_0 \, E_{b_S^\infty} \, \dot{r}_1\}.$$
If W_S^∞ is dense, then every $x \in \mathbb{R}$ is in A_p for some $p \in W_S^\infty$, and clearly $|A_p/E| = 1$. We say that p "captures" the E-class x/E if $|A_p/E| = 1$ and $A_p \cap x/E \neq \emptyset$. So all E-classes are captured and we can define
$$\phi_S : \mathbb{R}/E \to \mathbb{Q}_S^\infty \subset \kappa_S^\infty$$
by letting $\phi_S(x/E)$ be the least member of $\mathbb{Q}_s^\infty$ that captures x/E. This is clearly $\mathsf{OD}_{S,\mu}$.

If W_S^∞ is not dense, then by Łoś's theorem, we find a measure one set of σ on which this fact is true of W_S^σ. Fix σ and $p \in \mathbb{Q}_S^\sigma$ such that
$$\forall p' \leq_{\mathbb{Q}_S^\sigma} p \, \exists p_0, p_1 \leq_{\mathbb{Q}_S^\sigma} p' \, H_S^\sigma \models \text{“}(p_0,p_1) \Vdash \dot{r}_0 \not{E}_{S^\sigma} \dot{r}_1.\text{”}$$
We can enumerate the dense subsets of $\mathbb{Q}_S^\sigma$ in H_S^σ as $\{D_i : i \in \omega\}$ and use the above to build p_s for $s \in 2^{<\omega}$ so that for each $f \in 2^\omega$, $G_f = \{p_{f \restriction i} : i \in \omega\}$ generates a generic filter for H_S^σ with corresponding real r_f (in V) such that
$$H_S^\sigma[r_f, r_{f'}] \models r_f \not{E}_{S^\sigma} r'_f.$$
Recall that S^σ has the property that $E_{S^\sigma} = E_S$ on reals $\mathbb{Q}_S^\sigma$-generic over H_S^σ and thus we have that $r_f \not{E} \, r_{f'}$ for $f, f' \in 2^\omega$ with $f \neq f'$. This shows that E is not thin. □

4.2. The main theorem for $L(T, \mathbb{R})$.
Now we indicate how to generalize Theorem 4.8 to obtain Theorem 1.2 when $V = L(T, \mathbb{R})$ for $T \subseteq \mathsf{ORD}$. As in the proof of Theorem 4.8, we reduce to the case of an ∞-Borel pre-partial order $\leq_S$ on $\mathbb{R}$, and one needs only modify the definition of W_S^∞. The relevant set becomes
$$W_S^\infty = \{(p,p) \in \mathbb{Q}_S^\infty \times \mathbb{Q}_S^\infty : \Vdash^{H_S^\infty}_{\mathbb{Q}_S^\infty \times \mathbb{Q}_S^\infty} \dot{r}_0 \leq_{b_S^\infty} \dot{r}_1 \text{ or } \dot{r}_1 \leq_{b_S^\infty} \dot{r}_0\},$$
where $\leq_S$ is the ∞-Borel pre-partial order. If the set is not dense, just as before, we can find a copy of 2^ω consisting of $\leq_S$-pairwise incomparable elements. If the set is dense, then A_p is a pre-chain for $p \in W_S^\infty$, and every $x \in \mathbb{R}$ is in A_p for some such p.

4.3. The $\mathsf{AD}_\mathbb{R}$ case.
Assume AD^+ and $V = L(\mathcal{P}(\mathbb{R}))$ yet $V \neq L(T, \mathbb{R})$ for any $T \subseteq \mathsf{ORD}$. We begin by explaining how to obtain Theorem 1.4. As mentioned previously, the argument in this case was suggested by Woodin.

Given X, find some $\gamma < \Theta$ and $s_0 \in \gamma^\omega$ such that $X \in \mathsf{OD}_{s_0}$. This is possible, by Theorem 3.3.

The key idea is to define, for $\sigma \in [<\Theta]^\omega$,
$$X_{\sigma,\alpha} = \{a \in X : \exists t \in \mathbb{R} \, (a \text{ is definable from } \sigma, s_0, \alpha, t)\}.$$
The reason for relativizing to σ will become apparent soon. Notice that if $\sigma \subset \tau$ and $a \in \mathsf{OD}_{\sigma,s_0,t}$ for some t, then there is $t' \in \mathbb{R}$ so that $a \in \mathsf{OD}_{\tau,s_0,t'}$.

Let $E_{\sigma,\alpha}$ be the equivalence relation on $\mathbb{R}$ induced by $X_{\sigma,\alpha}$. If any $E_{\sigma,\alpha}$ is thick, then we are done. Otherwise, uniformly in α, there is an OD_{σ,s_0} ∞-Borel code $S_{\sigma,\alpha}$ for $E_{\sigma,\alpha}$ and a corresponding $\phi_{\sigma,\alpha} : \mathbb{R}/E_{\sigma,\alpha} \to \gamma_\alpha$ inducing $E_{\sigma,\alpha}$.

In particular (by the argument for the previous case) $X_{\sigma,\alpha} \subset \mathsf{OD}_{\sigma,s_0}$ and thus $X_\sigma \subset \mathsf{OD}_{\sigma,s_0}$, where $X_\sigma = \bigcup_\alpha X_{\sigma,\alpha}$. Let $<_\sigma$ be the OD_{σ,s_0} well-order of X_σ.

For each $\xi < \Theta$ let $X_\xi = \bigcup_{\sigma \in \mathcal{P}_{\omega_1}(\xi)} X_\sigma$, and notice for $\xi < \xi'$, $X_\xi \subseteq X_{\xi'}$.

Woodin's main observation here is that the supercompactness measures can be used to uniformly well-order the sets X_ξ and hence to obtain a well-order of X. Namely, set

$$a <_\xi a' \iff \forall^*_{\mu_\xi} \sigma\, [a <_\sigma a'].$$

This shows that $X_\xi \subset \mathsf{OD}_{s_0}$ and hence $X \subset \mathsf{OD}_{s_0}$. So X is well-orderable.

This argument can be easily modified so we also obtain Theorem 1.2. Namely, from the previous subsection, we can assume each $\leq \restriction X_\xi$ is a well-ordered union of pre-chains; this is uniform in ξ, and just as before we use the supercompactness measures to obtain that $\leq$ itself is a well-ordered union of pre-chains.

4.4. The E_0-dichotomy.

Finally, we sketch how to prove Theorem 1.3. The argument in Hjorth [**10**] greatly resembles the construction in Harrington-Kechris-Louveau [**6**] and the proof above, and we only indicate the required additions, and leave the details to the interested reader.

Assume AD^+ and that $V = L(T, \mathbb{R})$ for some $T \subset \mathsf{ORD}$, or else $V = L(\mathcal{P}(\mathbb{R}))$. Let $(X, \leq)$ be a partially ordered set. First, the techniques above and Theorem 2.5 of Hjorth [**10**] generalize straightforwardly to give us that, if X is a quotient of 2^ω by an equivalence relation E, then either there is an injection of $2^\omega/E_0$ into X whose image consists of pairwise $\leq$-incomparable elements, or else for some α there is a sequence

$$(A_\beta : \beta < \alpha)$$

such that for any $x, y \in \mathbb{R}$,

$$[x]_E \leq [y]_E \iff \forall \beta < \alpha\, (x \in A_\beta \to y \in A_\beta)$$

(just vary slightly the definition of $\mathcal{A}([\![f]\!]_\mu)$ in page 1202 of Hjorth [**10**]. See also Foreman [**4**] for a similar approach under a slightly stronger assumption; this approach can be transformed into a proof from AD^+ by using the AD^+-version of Solovay's basis theorem mentioned in page 14.)

Using this, Theorem 1.3 follows immediately, first for models of the form $L(T, \mathbb{R})$, just as in Theorem 2.6 of Hjorth [**10**], and then for models of $\mathsf{AD}_\mathbb{R}$ using the 'weaving together' technique from the previous subsection.

5. The countable-finite game in canonical models of AD^+

In this section we work in ZF and prove Theorem 1.8. We are interested in the countable-finite game in the absence of choice; here are some obvious observations:

FACT 5.1 (ZF). *Player II has a winning strategy in $CF(S)$ whenever S is countable or Dedekind-finite.*

PROOF. This is obvious if S is countable. Recall that S is Dedekind-finite iff $\omega \not\preceq S$. It follows that if S is Dedekind-finite, then each move of player I must be a *finite* set. $\square$

FACT 5.2 (ZF). *Assume every uncountable set admits an uncountable linearly orderable subset.*

Then Player I has a winning strategy in $CF(S)$ iff some uncountable subset of S is the countable union of countable sets.

PROOF. Suppose first that S admits an uncountable subset that can be written as a countable union of countable sets. We may as well assume that S itself admits such a representation, and that S is linearly orderable. It suffices to show that any countable union of finite subsets of S is countable. For this, let $<$ linearly order S, and let $(S_n : n \in \omega)$ be a sequence of finite subsets of S. We may as well assume they are pairwise disjoint. We can then enumerate their union $S^* = \bigcup_n S_n$ by listing the elements of each S_n in the order given by $<$, and listing the elements of S_n before those of S_m whenever $n < m$. This gives an ordering of S^* in order type at most ω.

Conversely, suppose any countable union of countable subsets of S is countable, and let F be a strategy for player I. Define a sequence $(C_n)_{n\in\omega}$ of subsets of S as follows:

- $C_0 = F(\langle\rangle)$,
- For $n > 0$, $C_n = \bigcup_{\{x_i\,:\,i<n\}\subseteq\bigcup_{i<n} C_i} F(\langle\{x_i\} : i < n\rangle)$.

By induction, each C_n is countable, and therefore so is $\bigcup_n C_n$. Using an enumeration of this last set, it is straightforward for player II to win a run of $CF(S)$ (by playing *singletons*) against player I following F. It follows that F is not winning for player I. □

From the argument above, we see that it is consistent that player I wins $CF(S)$ for some S. For example, player I wins $CF(\omega_1)$ whenever $\mathrm{cf}(\omega_1) = \omega$, and wins $CF(\mathbb{R})$ in the model of Feferman-Levy [**3**], where $\mathbb{R}$ is a countable union of countable sets.

From now on, assume that AD^+ holds and that $V = L(T, \mathbb{R})$ for some $T \subseteq \mathsf{ORD}$, or $V = L(\mathcal{P}(\mathbb{R}))$. The dichotomy Theorem 1.4 immediately gives the basis theorem for cardinalities, Corollary 1.5.

It follows that there are no infinite Dedekind-finite sets, and that (since ω_1 is regular) any countable union of countable sets is countable.

By the previous result on winning strategies for Player I, we now have:

COROLLARY 5.3. *Assume that AD^+ holds and that $V = L(T, \mathbb{R})$ for some $T \subseteq \mathsf{ORD}$, or $V = L(\mathcal{P}(\mathbb{R}))$. Then, for no set S, Player I has a winning strategy in $CF(S)$.* □

It remains to study when Player II has a winning strategy in $CF(S)$. We may assume that S is uncountable, and analyze the two possibilities $\omega_1 \preceq S$ and $\mathbb{R} \preceq S$ separately.

LEMMA 5.4 (ZF). *Assume $\omega_1 \not\preceq \mathbb{R}$. If $\omega_1 \preceq S$, then Player II has no winning strategy in $CF(S)$.*

Recall that AD implies that $\omega_1 \not\preceq \mathbb{R}$.

PROOF. From a winning strategy F for player II, we can find enumerations of all countable ordinals: Without loss, $\omega_1 \subseteq S$. Consider the run of the game where player I plays $\alpha, \alpha + 1, \alpha + 2, \ldots$. Then α is covered by the finite subsets of

α that player II plays by turns following F, and these finite sets provide us with an enumeration of α in order type ω. But it is trivial to turn such a sequence of enumerations into an injective ω_1-sequence of reals. □

LEMMA 5.5 (ZF). *Assume* $\mathsf{AC}_\omega(\mathbb{R})$ *and that there is a fine measure on* $\mathcal{P}_{\omega_1}(\mathbb{R})$. *If* $\mathbb{R} \preceq S$ *then Player II has no winning strategy in* $CF(S)$.

AD implies both that $\mathsf{AC}_\omega(\mathbb{R})$ holds, and that there is such a measure; the latter can be obtained, for example, by lifting Solovay's club measure on ω_1, or Martin's cone measure on the Turing degrees.

PROOF. We may assume $S = \mathbb{R}$. Assume player II has a winning strategy F. Fix a fine measure μ on $\mathcal{P}_{\omega_1}(\mathbb{R})$. We find a measure one set C such that player II always plays the same (following F) for any valid play of player I using members of C. Since C is uncountable, this shows that player I can defeat F, contradiction.

Notice that we can identify $\mathcal{P}_\omega(\mathbb{R})$ with $\mathbb{R}$. Using the σ-completeness of μ, there is a measure 1 set A_0 and a fixed finite set T_0 such that for all $\sigma \in A_0$, $F(\langle\sigma\rangle) = T_0$. To see this, notice that (identifying T_0 with a real) for each $i \in \omega$ there is a unique $j_i \in \omega$ and a measure 1 set A_0^i such that if $\sigma \in A_0^i$ then $F(\sigma)(i) = j_i$, and we can set $A_0 = \bigcap_i A_0^i$.

Similarly, there is a measure 1 set $A_1 \subset A_0$ and a fixed finite set T_1 such that for all $\sigma, \sigma' \in A_1$ with $\sigma' \supseteq \sigma$, $F(\langle\sigma, \sigma'\rangle) = T_1$.

Continue this way to define sets $A_0, A_1, \dots$ Let $A = \bigcap_i A_i$. Then A has measure 1. In particular, $\bigcup A$ is uncountable. However, for any $\sigma_0 \subseteq \sigma_1 \subseteq \dots$ with all the σ_i in A, $F(\langle\sigma_0, \dots, \sigma_i\rangle) = T_i$. Since $\bigcup_i T_i$ is countable, we can find r, σ with $r \in \sigma$, $\sigma \in A$, $r \notin \bigcup_i T_i$, and from this it is straightforward to construct a run of $CF(\mathbb{R})$ where player I defeats player II following F, and so F was not winning after all. □

From the basis theorem, Corollary 1.5, we now have:

COROLLARY 5.6. *Assume that* AD^+ *holds and that* $V = L(T, \mathbb{R})$ *for some* $T \subseteq \mathsf{ORD}$, *or* $V = L(\mathcal{P}(\mathbb{R}))$. *Then, for no uncountable set* S, *Player II has a winning strategy in* $CF(S)$. □

Combining this with the previous result for Player I, we conclude:

COROLLARY 5.7. *Assume that* AD^+ *holds and that* $V = L(T, \mathbb{R})$ *for some* $T \subseteq \mathsf{ORD}$, *or* $V = L(\mathcal{P}(\mathbb{R}))$. *Then, for all uncountable sets* S, $CF(S)$ *is undetermined.* □

6. Questions

Recall that the main step of the dichotomy proof consists of passing from an ∞-Borel code S to a local version S^σ which correctly computes A_S on suitable inner models, N_σ, that satisfy choice and, moreover, this computation is preserved by passing to forcing extensions of N_σ.

QUESTION 6.1. Does our analysis extend to models of the form $L(\mathcal{P}(\mathbb{R}))[X]$ for sets X that satisfy some suitable form of condensation, so that Theorem 1.2 holds for these models as well?

Vaguely, the point is that we need enough absoluteness of the structure so that the process of passing to countable structures and then taking an ultrapower provides us with the appropriate ∞-Borel codes.

In a different direction, one can ask:

QUESTION 6.2. To what extent can we recover the local bounds on the witnessing ordinals known previously in particular cases of Theorems 1.2, 1.4, and 1.3?

For example, it is not too difficult to combine our analysis with known techniques, to see that, as in Harrington-Marker-Shelah [**7**], a thin Borel partial order is a *countable* union of chains, or that quotients of $\mathbb{R}$ by projective equivalence relations can be well-ordered *in type less than* $\underset{\sim}{\delta}^1_n$ for an appropriate n, as shown in Harrington-Sami [**8**]. But it seems that, in general, the passing to ultrapowers blows up the bounds beyond their expected values.

Let $\mathfrak{c} = |\mathbb{R}|$. Under determinacy, $\omega_1 + \mathfrak{c}$ is an immediate successor of $\mathfrak{c}$. It is a known consequence of $\mathsf{AD}_\mathbb{R}$ (see Ditzen [**2**]) that $|2^\omega/E_0|$ is also an immediate successor of $\mathfrak{c}$; in fact, any cardinal strictly below $|2^\omega/E_0|$ injects into $\mathfrak{c}$. Let

$$S_1 = \{a \in \mathcal{P}_{\omega_1}(\omega_1) : \sup(a) = \omega_1^{L[a]}\}.$$

In Woodin [**25**] it is shown, under $\mathsf{ZF}+\mathsf{DC}+\mathsf{AD}_\mathbb{R}$, that $|S_1|$ is yet another immediate successor of $\mathfrak{c}$.

On the other hand, in $\mathsf{ZF} + \mathsf{AD}^+ + \neg\mathsf{AD}_\mathbb{R}$, Woodin [**25**] shows that there is at least one cardinal intermediate between $\mathfrak{c}$ and $|S_1|$, and there is also at least one cardinal intermediate between $\mathfrak{c}$ and $\mathfrak{c}\cdot\omega_1$ incomparable with ω_1. We do not know of a complete classification of immediate successors of $\mathfrak{c}$ under our working assumptions, or whether this is even possible.

QUESTION 6.3. Is it possible to classify, under $\mathsf{AD}^+ + V = L(\mathcal{P}(\mathbb{R}))$, the immediate successors of $|\mathbb{R}|$?

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