

Comments and corrections to “*An introduction to core model theory*,” by Benedikt Löwe and John Steel

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The paper appeared in **Sets and Proofs, Logic Colloquium 1997**, vol 1, London Mathematical Society Lecture Notes 258, Cambridge University Press, Cambridge, 1999, 103–157. The numbering follows the version posted online at Steel’s website (which coincides with the published version as far as I could tell).

- In page 18, l-2, $Q = M_\Omega^U$
- In page 19, $a \notin i_{\kappa,\sigma}^U(A_\kappa)$ and $a \notin i_{\kappa,\nu}^U(A_\kappa)$
- In page 21, lemma 2.13, M is W .
- In page 22, def. 2.14, \leq should be \geq
- In page 23, Theorem 2.16 (2), $[0, \alpha]_T$; (3), $\pi(\eta)$
In the proof, π_β is σ_β
- In page 25, the definitions of η_0^1 and η_1^1 should be exchanged.
Corollary 2.18 should assume that M is iterable with a unique (generalized) iteration strategy.
- In page 26, l.2, \leq should be $<$
- In page 29, proof, l 4, κ is an uncountable *regular* cardinal
- In page 40, proof of lemma 3.5. Say that the extender E witnessing measurability comes from $N_{\alpha+1} = (J_{\eta+1}[\vec{E}], \in, \vec{E}, \tilde{E})$, possibly after taking some cores. Then $\text{crit}(\tilde{E})$ is inaccessible in V by definition 2.21. If at some stage a core is taken which collapses $\text{crit}(\tilde{E})$ to some smaller ρ , then E would not be total. Hence, $E = \tilde{E}$ and $\text{crit}(E)$ is inaccessible.

Of course, we are assuming the hypothesis of lemma 3.5 really talks about the first cardinal *witnessed by the K^c sequence* to be measurable. For more on this, see the footnote in page 47.

The argument, as given, looks weaker than it really is. It looks like 3.5 holds in virtue of the superfluous clause in definition 2.21 asking κ to

be inaccessible. Weaker conditions in the background certificates can be asked, but all seem to require some form of countable completeness of the background extenders. This implies that their critical points must be uncountable, so the proof of Proposition 3.6 still works.

- In page 41, proof of Prop. 3.8. This is not the standard notation. It is easy to see that A_0 is stationary in Ω , but the “first Mahlo derivative” of a set of μ_0 -measure 1 has μ_0 -measure 1, unlike A_0 .
- In page 42, definition 3.9, condition 2. Call the set Γ_α . The condition states that Γ_α is unbounded in α^+ and whenever β is a limit point of Γ_α , $\text{cf}(\beta) = \alpha$, then $\beta \in \Gamma_\alpha$.

1-3, K^c (cf def. 3.20)

footnote 80. In fact, if W is universal and Γ is thick in W , $\mathcal{H}^W(\Gamma)$ (the collapse of the hull of Γ inside W) is also universal, by definition of A_0 and the remark after theorem 3.4.

- In page 43, $\pi : N \rightarrow N_\alpha$ must be the *inverse of the* transitive collapse.
- In page 44, line 1, to see H and N are universal, see footnote 80.
1-6 of proof, (since N has the hull property at μ)
- In page 45, 1-1. (or else, $\tau^{K^c}[a^*, d] \in D_{K^c}$, contradicting $D_{K^c} \subseteq b_0 \leq b_\xi$.)
- In page 47, 1. -2 before Lemma 3.27, K is very sound (cf [St96, Lemma 5.9])

Add the following footnote after the first W in l. 6 of proof of Lemma 3.27: i.e., the extender in the sequence with $\text{crit} = \alpha$ and smallest index. We have to verify 1) If $W \models \alpha$ measurable, then there is an extender in the W sequence witnessing this. 2) The smallest such extender (in terms of index) is of order 0 (a priori, the trivial completion of $E \upharpoonright \alpha^+$, the natural candidate, where E is the smallest extender in the sequence with $\text{crit} = \alpha$, may not be in the sequence but in some ultrapower.

For the purpose of the argument at hand, this is not necessary, though: just call “the order zero total measure on α ” the extender in the sequence with smallest index, measuring all the subsets of α . After taking the ultrapower, no extender in the sequence witnesses this (or, even more relevant to the argument at hand, let’s only care about measurable cardinals witnessed by an extender in the sequence).

For those measurable cardinals, we need the following:

LEMMA. Let W be a weasel, $W \models \alpha$ measurable, and there is an extender witnessing it in the sequence. ($W \models \text{ZFC}$ should be enough, we do not need $OR^W = \Omega$). Let γ be least such that E_γ^W has critical point α and measures all the subsets of α . Then $o^{\bar{E}^W}(E_\gamma^W) = 0$, i.e., no extender

in the sequence of the ultrapower by E_γ^W has critical point α (and is total on α).

PROOF. Consider the ultrapower embedding $\pi : W \rightarrow \tilde{W} = \text{Ult}(W, E_\gamma^W)$. By the coherence conditions 1,2 in def. 2.1, no extender on \tilde{W} has critical point α , index $\leq \gamma$, and is total on (subsets of) α . Thus, if an extender E on the sequence of \tilde{W} has critical point α , its index is $> \gamma$. So we can look at $E \upharpoonright \alpha^+$. Either it is in the \tilde{W} sequence, with index $< \gamma$ (a contradiction), or $\vec{E}_{\alpha^+}^{\tilde{W}} \neq \emptyset$ (and in the ultrapower of \tilde{W} by $\vec{E}_{\alpha^+}^{\tilde{W}}$, the trivial completion of $E \upharpoonright \alpha^+$ is in the sequence). But this is impossible, since α^+ is a cardinal of \tilde{W} , but no extender with index a cardinal is active (cf. the last remark in chapter 1 of [MiSt94] or the comments after the statement of Thm 1.1 of [Sch95]). QED

REMARK. If W is a universal weasel, $W \models$ “I am iterable” and $W \models$ “ α is measurable,” then there is a witness to the measurability of α in the extender sequence of W .

PROOF (Sketch). Compare W and \tilde{W} , where \tilde{W} is the ultrapower of W by a measure on α . The argument to follow uses a bit more: We actually assume $W \models V = K$ (as $K^K = K$ if $K \models$ “I am iterable” and K elementary embeds in W by Theorem 3.23. We can actually dispense with this extra assumption.)

Working in W , find W_1 witnessing the very soundness of a sufficiently large initial segment of W , so $W_1 \models \alpha$ is measurable.

Let $W_2 = \text{Ult}(W_1, \mu)$ where μ is a measure on α . W_1 has the definability property up to $\alpha + 1$ (at least), but W_2 does not have it at α (this is easy to check by the arguments in this paper).

Now compare W_1 and W_2 . They iterate to a common model W^* . Let $\pi : W_1 \rightarrow W_2$ be the natural embedding, $\pi_1 : W_1 \rightarrow W^*$, $\pi_2 : W_2 \rightarrow W^*$ the iterations arising from the comparison. Then $\text{crit}(\pi_i) \geq \alpha$, $i = 1, 2$. As usual, we can get $\text{crit}(\pi_1) = \text{crit}(\pi_2) = \alpha$. But then there is an extender on α used in the iteration.

To finish the proof we need to notice that no extender overlapping α could have been used before reaching the model in which the extender on α lives, or by the compatibility of the models present in an iteration tree we would not have had $\mathcal{P}(\alpha)$ unchanged. So in some model there is an extender on α , thus there is one of order zero by the LEMMA and, by compatibility again, that extender also had to be present in W_1 and therefore in W . QED

- In page 49, Theorem 3.32 needs to assume that $\rho_\omega^M \geq \alpha$, α a cardinal of M .

Proof, 1. 2. Recall that every proper initial segment of K is very sound. Thus, W exists.

l. 4. After iterable, add: Since α is a K -cardinal and $J_\alpha^K = J_\alpha^W$, α is a W -cardinal as well: Let V' witness the very soundness of $J_{\alpha^+}^K$ where α^+ is computed in the sense of K . Compare V' and W , getting maps $i : V' \rightarrow Q$ and $j : W \rightarrow Q$ for some common model Q . If $\text{crit}(i)$ or $\text{crit}(j) < \alpha$ then $\text{crit}(i) = \text{crit}(j)$ using the fact that the hull property fails in Q at α . Using the definability and hull properties up to α for V' and W we get the usual contradiction that the first extenders used on i and j are compatible. Thus, $\text{crit}(i) \geq \alpha$, $\text{crit}(j) \geq \alpha$. But then $\mathcal{P}(\alpha)^{V'} = \mathcal{P}(\alpha)^Q = \mathcal{P}(\alpha)^W$, and α is a cardinal in W .

l. -1. See [St96, Theorem 6.11] for details.

- In page 54, NeeSt9? is NeeSt99, vol64 (1999) 1285–1294.