On Real-valued measurability and Lebesgue measurable sets

Andrés Eduardo Caicedo*

October 11, 2005

Abstract

We show that the existence of real-valued measurable cardinals does not settle the range of Lebesgue measure on the projective sets.

For κ a cardinal, let $\mathsf{RVM}(\kappa)$ denote the statement that κ is real-valued measurable. In [2] it is shown that the generic invariance of the class $(\Sigma^2_2)^+$ with respect to real-valued measurability is not a theorem of ZFC, even in the presence of projective absoluteness (see [2, §3] for definitions and notation). In this note we explain why the mention of projective absoluteness is necessary to avoid trivialities.

The existence of real-valued measurable cardinals implies that all reals have sharps and therefore all $\utilde{\mathbf{\Sigma}}^1_2$ sets are Lebesgue measurable. If in addition $\kappa \leq \mathfrak{c}$, we say that κ is atomlessly measurable.

Fact 1. *Assume that there are atomlessly measurable cardinals. Then all $\utilde{\mathbf{\Delta}}^1_3$ sets are Lebesgue measurable.*

Proof. By [1, Theorem 9.4.6], if $\utilde{\mathbf{\Sigma}}^1_2$ sets are Lebesgue measurable then for any λ, $V^{\mathtt{Random}_\lambda} \models$ $\utilde{\mathbf{\Delta}}^1_3$ sets are Lebesgue measurable, where $\mathtt{Random}_\lambda$ is the standard forcing for adding λ many random reals. This is a projective statement about $V^{\mathtt{Random}_\lambda}$ by [1, Lemma 9.1.2]. But it is an easy consequence of Solovay's characterization of real-valued measurability in terms of generic elementary embeddings [2, Theorem 1.6] that if there are atomlessly measurable cardinals, then for some $\lambda \geq \aleph_1$, V and $V^{\mathtt{Random}_\lambda}$ coincide about projective statements. □

Fact 1 cannot be improved in general:

Example 2. *Suppose $V = L[\mu]$ is the smallest inner model with a measurable cardinal κ or, more generally, a model with a measurable cardinal κ in which the reals admit a* good Δ^1_3 well-ordering. *Then* $V^{\mathtt{Random}_\kappa} \models \kappa = \mathfrak{c}$, $\mathsf{RVM}(\kappa)$, and there is a nonmeasurable Σ^1_3 set.

*Department of Mathematics, Mail code 253-37, California Institute of Technology, Pasadena, CA 91125
E-mail : caicedo@caltech.edu

Proof. Recall that the reals admit a good Δ^1_3 well-ordering iff there is a Δ^1_3 well-ordering $\leq$ of $\mathbb{R}$ of length ω_1 such that the set of reals coding proper initial segments of $\leq$ is a Σ^1_3 set. Every real in $V^{\mathtt{Random}_\kappa}$ is added to V by small forcing (in fact, by a forcing isomorphic to $\mathtt{Random}_\omega$), and therefore V and $V^{\mathtt{Random}_\kappa}$ agree about Σ^1_3 statements. Since $\mathtt{Random}_\kappa$ is ccc and therefore preserves ω_1, by arguing about the Δ^1_3 well-ordering in V as in [5, Theorem 3.28] it follows that $\mathbb{R}^V$ is Σ^1_3 in $V^{\mathtt{Random}_\kappa}$. But $\mathbb{R}^V$ is not Lebesgue measurable as it does not have measure zero, arguing as in [1, Lemma 3.2.39], but it has size $\omega_1 < \mathfrak{c}$ and therefore cannot have positive measure. Finally, by Solovay's well-known result (see, for example, [3, Theorem 2C]) $\kappa = \mathfrak{c}$ and $\mathsf{RVM}(\kappa)$ hold in $V^{\mathtt{Random}_\kappa}$. It follows that the existence of atomlessly measurable cardinals does not imply that Σ^1_3 sets are Lebesgue measurable. □

On the other hand, $T =$"There exists an atomlessly measurable cardinal and all projective sets are Lebesgue measurable" is equiconsistent with the existence of measurable cardinals. This is because by a well-known result of Kunen (see [4]), if λ is weakly compact then the projective theory of $V^{\mathtt{Coll}(\omega,<\lambda)}$ is frozen under ccc forcing, and by Solovay's theorem (see [1]) all projective sets are Lebesgue measurable in this model. If κ is measurable and $\kappa > \lambda$, then $V^{\mathtt{Coll}(\omega,<\lambda)*\mathtt{Random}_\kappa}$ is a model of T.

References

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