

# Regressive functions on pairs

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October 23, 2007

## Abstract

We compute an explicit upper bound for the regressive Ramsey numbers by a combinatorial argument, the corresponding function being of Ackermannian growth. For this, we look at the more general problem of bounding  $g(n, m)$ , the least  $l$  such that any regressive function  $f : [m, l]^{[2]} \rightarrow \mathbb{N}$  admits a min-homogeneous set of size  $n$ . Analysis of this function also leads to the simplest known proof that the regressive Ramsey numbers have rate of growth at least Ackermannian. Together, these results give a purely combinatorial proof that, for each  $m$ ,  $g(\cdot, m)$  has rate of growth precisely Ackermannian, considerably improve the previously known bounds on the size of regressive Ramsey numbers, and provide the right rate of growth of the levels of  $g$ . For small numbers we also find bounds on their value under  $g$  improving the ones provided by our general argument.

**2000 MSC:** 05D10 (03D20)

**Keywords:** Kanamori-McAloon theorem, regressive Ramsey numbers, Ackermann's function.

## 1 Introduction

Throughout this paper,  $\mathbb{N} = \{0, 1, \dots\}$ . For  $1 \leq n, k \leq m$ , let  $m \rightarrow (n)_{reg}^k$  be the following assertion:

Whenever  $f : [1, m]^{[k]} \rightarrow [0, m-k]$  is regressive, there is  $H \in [1, m]^{[n]}$  min-homogeneous for  $f$ .

Similarly, for  $X \subseteq \mathbb{N}$  infinite, let  $X \rightarrow (\mathbb{N})_{reg}^k$  mean that for every regressive  $f : X^{[k]} \rightarrow \mathbb{N}$  there is  $H \subseteq X$  infinite and min-homogeneous for  $f$ . Here,

- $X^{[k]}$  is the collection of  $k$ -sized subsets of  $X$ .

- $f : X^{[k]} \rightarrow \mathbb{N}$  is regressive iff  $f(s) < \min(s)$  whenever  $s \in X^{[k]}$  and  $\min(s) > 0$  (where  $\min(s)$  is the least element of  $s$ ).
- For such an  $f$ ,  $H \subseteq X$  is min-homogeneous for  $f$  iff  $0 \notin H$  and, whenever  $s, t \in H^{[k]}$  and  $\min(s) = \min(t)$ , then  $f(s) = f(t)$ .
- $[n, m] = \{n, n+1, \dots, m\}$ . Similarly for other interval notation.

The following is the main result of Kanamori-McAloon [5]:

**Theorem 1.1.** 1. For any  $k, n \in \mathbb{N}$ , there is  $m$  such that  $m \rightarrow (n)_{reg}^k$ .

2. Item 1 is not a theorem of Peano Arithmetic PA.

In fact, in Kanamori-McAloon [5] a level-by-level correspondence is established between the values of  $k$  and the amount of induction required to prove the existence of the function that to  $n$  assigns the least  $m$  as in Theorem 1.1.1; see Carlucci-Lee-Weiermann [2] for more on this.

In this paper I only deal with  $k = 2$  although, in Section 3, I present a short proof of Theorem 1.1.1. In Section 4, I show that

$$g(n) = \text{least } l \text{ such that } l \rightarrow (n)_{reg}^2$$

is provably total in PA. In fact, I provide an explicit (recursive) upper bound for  $g(n)$ , thus showing by purely elementary means that its rate of growth is at most Ackermannian.

To state the result, let  $g(n, m)$  be the least  $l$  such that for any regressive  $f : [m, l]^{[2]} \rightarrow [0, l-2]$ , there is a min-homogeneous set for  $f$  of size  $n$ . (From now on, all mentions of  $g$  refer to this two-variable function.) Clearly  $g(n, m) \leq g(n, m+1)$ ,  $g(2, m) = m+1$  and, by the pigeonhole principle,  $g(3, m) = 2m+1$ . Also, let  $G(n, m)$  be the least  $l$  such that for any regressive  $f : [m, l]^{[2]} \rightarrow [0, l-2]$ , there is a min-homogeneous set for  $f$  of size  $n$  whose minimum element is  $m$ . It may not be immediate that  $G$  is well defined, but this is addressed by Remark 3.3 and the proof of Theorem 4.1. Thus,  $G(2, m) = g(2, m)$ ,  $G(3, m) = g(3, m)$ ,  $G(n+1, 1) = g(n+1, 1) = g(n, 2)$  and in general  $g(n, m) \leq G(n, m)$ . Set  $g^0(n, m) = m$  and  $g^{k+1}(n, m) = g(n, g^k(n, m))$ . We then have:

**Theorem 1.2.** 1.  $G(4, m) = 2^m(m+2) - 1$ .

2. Let  $\alpha_{-1} = 0$  and for  $0 \leq i < m$  let  $d_i = g^i(4, m+1)$  and  $\alpha_i = (\alpha_{i-1} + m + 3 + i)(2^{d_i} - 1)$ . Then  $g(5, m) \leq (2m+1) + \sum_{i=0}^{m-1} \alpha_i$ .

3. For all  $n$ , there is a constant  $c_n$  such that  $G(n, m) < A_{n-1}(c_n m)$  for almost all  $m$ .

Here,  $A_n = A(n, \cdot)$  where  $A$  is Ackermann's function, see Section 2. Theorem 1.2.2 is shown by adapting the proof of Blanchard [1, Lemma 3.1] (that bounds  $g(5, 2)$ ) to the more general problem of bounding  $g(5, m)$ . In Kojman-Shelah [7], explicit lower bounds for  $g$  are computed, showing that  $g$  is at least of Ackermannian growth (our notion of "Ackermannian growth" is more restrictive

than that of Kojman-Shelah [7] or Kojman-Lee-Omri-Weiermann [6], and is discussed in Section 2). In Section 5, I find lower bounds for  $G(n, m)$  and  $g(n, m)$  in terms of iterates of  $g(n - 1, \cdot)$ , and conclude:

**Theorem 1.3.**  $g(n, m) \geq A_{n-1}(m - 1)$  for all  $n \geq 2$ .

The proof of Theorem 1.3 is simpler and shorter than the proofs of lower bounds in Kojman-Shelah [7] and Kojman *et al.* [6] and increases these bounds significantly. Thus, the results of Sections 4 and 5 combine to give a very accessible and purely combinatorial proof of the result obtained in Kanamori-McAloon [5] by model theoretic methods, that  $g$  is not provably total in Primitive Recursive Arithmetic PRA but is “just shy” of it; in fact, the argument gives that, for each  $m$ , the function  $g(\cdot, m)$  has Ackermannian rate of growth. These results also establish the rate of growth of the function  $g(n, \cdot)$  as being precisely that of the  $(n - 1)^{\text{st}}$  level of the Ackermann hierarchy of fast growing functions.

In the literature, the values of  $g$  (more precisely, the values of  $g(\cdot, 2)$ ) are referred to as “regressive Ramsey numbers.” In Section 6, I improve the upper bound for  $g(4, m)$  and show:

**Theorem 1.4.**  $g(4, 3) = 37$ .

I also improve the upper bound for  $g(4, 4)$  provided by the general argument of Section 6. The figures so obtained improve the previously known bounds for small regressive Ramsey numbers obtained in Blanchard [1] and Kojman *et al.* [6].

I occasionally abuse notation by writing  $f(t_1, t_2)$  for  $f(t)$  where  $t_1 < t_2$  and  $t = \{t_1, t_2\}$ .

## 2 Preliminaries on Ackermannian functions

In this section I collect several standard results about Ackermannian growth; notice that the notion I use is more restrictive than the version used in Kojman-Shelah [7] or Kojman *et al.* [6], where a function is called Ackermannian simply if it eventually dominates each primitive recursive function.

**Definition 2.1.** Given functions  $g, h : \mathbb{N} \rightarrow \mathbb{N}$ , say that  $h$  *eventually dominates*  $g$ , in symbols  $g <_* h$ , iff  $g(m) < h(m)$  for all but finitely many values of  $m$ .

**Definition 2.2.** Ackermann’s function  $A : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$  is defined by double induction as follows:

- $A(0, m) = m + 1$ .
- $A(n, 0) = A(n - 1, 1)$  for  $n > 0$ .
- $A(n, m) = A(n - 1, A(n, m - 1))$  for  $n, m > 0$ .

Let  $\text{Ack}(n) = A(n, n)$  and  $A_n = A(n, \cdot)$ . Sometimes in the literature it is Ack that is referred to as Ackermann's function. This is the standard example of a recursive but not primitive recursive function. The version presented above is due to Rafael Robinson and Rózsa Péter, see Robinson [8]. Notice that  $A_1(m) = m + 2$ ,  $A_2(m) = 2m + 3$ ,  $A_3$  has exponential rate of growth and  $A_4$  grows like a tower of exponentials.

**Definition 2.3.** Let  $f_0(m) = m + 1$  and  $f_{n+1}(m) = f_n^m(m)$  where the superindex indicates that  $f_n$  is iterated  $m$  times. Continue this hierarchy by letting  $f_\omega(m) = f_m(m)$  and  $f_{\omega+1}(m) = f_\omega^m(m)$ .

Notice that what in Kojman *et al.* [6] is called Ackermann's function is the map  $A'(n, m) = f_{n-1}(m)$ .

**Definition 2.4.** A function  $f : \mathbb{N} \rightarrow \mathbb{N}$  is (precisely) of Ackermannian growth if and only if there are constants  $c, C > 0$  such that for all but finitely many  $m$ ,  $f_\omega(cm) \leq f(m) \leq f_\omega(Cm)$ .

Similarly, say that a function's rate of growth is like that of the  $n^{\text{th}}$  level of the Ackermann hierarchy if there are constants  $c, C > 0$  such that for all but finitely many  $m$ ,  $A_n(cm) \leq f(m) \leq A_n(Cm)$ .

(Compare with Graham-Rothschild-Spencer [4, Section 2.7], where the relevant notion is called *Ackermannic*.)

The following two lemmas are standard and collect together several folklore results; see for example Graham-Rothschild-Spencer [4] and Cori-Lascar [3].

**Lemma 2.5.** 1. For all  $n$ ,  $A_n < A_{n+1}$  and  $f_n <_* f_{n+1}$ . In fact, for any  $C > 0$  and almost all  $m$ ,  $A_n(Cm) < A_{n+1}(m)$  for  $n > 0$ , and  $f_n(Cm) < f_{n+1}(m)$  for all  $n$ .

2. For all  $n > 0$ ,  $A_{n+1} <_* f_n$  and  $f_n(m) < A_{n+1}(cm)$  for some constant  $c = c_n$  and all  $m$ .

3.  $f_\omega$  and Ack are of Ackermannian growth.  $\square$

More precise quantitative versions of the above are possible, but Lemma 2.5 as stated suffices for our purposes.

**Lemma 2.6.** 1. If  $f$  is of Ackermannian growth, it eventually dominates each primitive recursive function. In particular, it eventually dominates each  $f_n$ .

2. If  $f$  is of Ackermannian growth then it is eventually dominated by  $f_{\omega+1}$ .

3. There is a function  $f$  that eventually dominates each  $f_n$  and is eventually dominated by  $f_{\omega+1}$  but is not of Ackermannian growth.

4. If  $g, h$  are strictly increasing primitive recursive functions and  $f$  is of Ackermannian growth, then so is  $g \circ f \circ h$ .  $\square$

### 3 Regressive functions

I start by proving the infinite version of Theorem 1.1.1. This is also done in Kanamori-McAloon [5], but the argument to follow is easier (in Kanamori-McAloon [5] this is accomplished using the Erdős-Rado canonization theorem). The proof of Theorem 1.2 in Section 4 was obtained by trying to produce a finitary and effective version of this argument for  $k = 2$ .

**Lemma 3.1.** *If  $X \subseteq \mathbb{N}$  is infinite, then for any  $k$ ,  $X \rightarrow (\mathbb{N})_{reg}^k$ .*

*Proof.* Let  $f : X^{[k]} \rightarrow \mathbb{N}$  be regressive. Without loss,  $k > 1$ . Define a decreasing sequence of infinite subsets of  $X$ ,  $X \setminus \{0\} = H_0 \supset H_1 \supset H_2 \supset \dots$  such that letting  $m_n = \min H_n$ , then  $(m_n)_{n \geq 0}$  is strictly increasing, as follows: Given  $H_n$ , let  $\varphi : (H_n \setminus \{m_n\})^{[k-1]} \rightarrow [0, m_n - 1]$  be the function  $\varphi(s) = f(\{m_n\} \cup s)$ . By Ramsey's theorem, there is  $H_{n+1}$  infinite and homogeneous for  $\varphi$ .

Then  $\{m_n : n \in \mathbb{N}\}$  is min-homogeneous for  $f$ .  $\square$

Theorem 1.1.1 follows now from a standard compactness argument:

**Corollary 3.2.**  $\forall n \forall k \exists l (l \rightarrow (n)_{reg}^k)$ .

*Proof.* Fix  $n$  and  $k$  counterexamples to the corollary. For each  $m \geq n, k$  it follows that there are regressive functions  $f : [1, m]^{[k]} \rightarrow [0, m - k]$  without min-homogeneous sets of size  $n$ . Consider the collection  $\mathcal{T}$  of all these functions, ordered by extension: Given  $f_1, f_2 \in \mathcal{T}$ ,  $f_1 : [1, m_1]^{[k]} \rightarrow [0, m_1 - k]$ ,  $f_2 : [1, m_2]^{[k]} \rightarrow [0, m_2 - k]$ , set  $f_1 < f_2$  iff  $m_1 < m_2$ , and  $f_2 \upharpoonright [1, m_1]^{[k]} = f_1$ . Then  $(\mathcal{T}, <)$  is an infinite finitely branching tree so, by König's lemma, it has an infinite branch. The functions along this branch fit together into a regressive function  $f : \mathbb{N}^{[k]} \rightarrow \mathbb{N}$  which contradicts Lemma 3.1 since it does not even admit min-homogeneous sets of size  $n$ .  $\square$

**Remark 3.3.** Notice that using this argument one can easily show that  $G(n, m)$  is well defined. The argument next section will also show this.

### 4 An Ackermannian upper bound for $G$

Here I prove Theorem 1.2.3; the argument resembles the "color focusing" technique from Ramsey theory.

**Theorem 4.1.** *For each fixed  $m$ ,  $G(n, m)$  is bounded by a function of Ackermannian growth. In particular, so is  $g(n, 2) \leq G(n, 2)$ .*

*Proof.* I find an upper bound for the function  $G(n, \cdot)$  by induction on  $n$ . In order to do this I introduce numbers  $s_i = s(i, n, m)$  for all  $n \geq 4$ ,  $m \geq 2$ , and  $1 \leq i \leq m$ , and argue that  $G(n, m) \leq s(m, n, m)$ .

Fix  $n \geq 4$ . The numbers  $s_i$  are computed in terms of the function  $G(n-1, \cdot)$ . Fix  $m$ , which we may assume is at least 2.

Define  $s(1, n, m), \dots, s(m, n, m)$  and  $t_0, t_1, \dots, t_{m-1}$  recursively as follows.

- Let  $t_0 = m + 1$ .
- Let  $s_1 = g(n - 1, t_0)$  and for  $1 \leq i < m$ , let  $s_{i+1} = G(n - 1, t_i)$ .
- For  $1 \leq j \leq m$  let  $B_j^{n,m} = B_j = \bigcup_{i=1}^j [t_{i-1}, s_i]$ , and denote by  $\prod B_j$  the Cartesian product  $\prod_{i \in B_j} [0, i - 1]$ .
- For  $1 \leq j < m$ , let  $t_j = (j + 1) \times |\prod B_j|$ .

We claim that  $G(n, m) \leq s(m, n, m)$ . To see this, suppose a regressive function  $f : [m, s_m]^{[2]} \rightarrow [0, s_m - 2]$  is given.

Fix  $j$ ,  $1 < j \leq m$ . Suppose  $f(m, \cdot) \upharpoonright B_j$  takes at most  $j$  values (This holds trivially for  $j = m$ .) We claim that either there is a min-homogeneous set for  $f$  of size  $n$  contained in  $\{m\} \cup B_j$  whose minimum element is  $m$ , or else  $f(m, \cdot) \upharpoonright B_{j-1}$  takes at most  $j - 1$  values.

Consider the regressive function

$$\psi : [t_{j-1}, s_j]^{[2]} \rightarrow [0, s_j - 2]$$

given by

$$\psi(u) = \begin{cases} f(u) & \text{if } u_1 > t_{j-1}, \\ \langle f(l, u_2) : l \in \{m\} \cup B_{j-1} \rangle & \text{if } u_1 = t_{j-1}, \end{cases}$$

where  $\langle \dots \rangle$  is a bijection from the Cartesian product  $C_j \times \prod B_{j-1}$  onto  $[0, t_{j-1}]$ , where  $C_j \subset [0, m - 1]$  has size  $j$  and contains the possible values that  $f(m, \cdot) \upharpoonright B_j$  can take.

Then (by definition of  $s_j$ ) there is a set  $\{a_1, \dots, a_{n-2}\} \subseteq [t_{j-1} + 1, s_j]$  min-homogeneous for  $f$  and such that for all  $k \in \{m\} \cup B_{j-1}$ ,  $\{k, a_1, \dots, a_{n-2}\}$  is also min-homogeneous for  $f$ . Let  $f(m, a_1) = c$ . If  $f(m, k) = c$  for any  $k \in B_{j-1}$ , then  $\{m, k, a_1, \dots, a_{n-2}\}$  is the min-homogeneous set we are looking for. Otherwise,  $f(m, \cdot) \upharpoonright B_{j-1}$  takes at most  $j - 1$  values, as claimed.

There is therefore no loss in assuming that  $f(m, \cdot) \upharpoonright B_1$  is constant but then by definition of  $s_1$  there is  $\{a_1, \dots, a_{n-1}\} \subseteq B_1$  min-homogeneous for  $f$ . Then  $\{m\} \cup \{a_1, \dots, a_{n-1}\}$  is also min-homogeneous, and we are done.

Define a function  $H(n, m)$  as follows:  $H(n, \cdot) = G(n, \cdot)$  for  $n \leq 4$  (see also Fact 5.3 below); in the argument above, let  $s'_i$  be the function resulting from replacing  $G(n - 1, \cdot)$  with  $H(n - 1, \cdot)$  in the definition of  $s_i$ , and let  $H(n, m) = s'(m, n, m)$ , so clearly  $G \leq H$ . It is easy to see, using standard arguments (or consider the proof of Theorem 1.2.3 below) that  $n \mapsto H(n, m)$  (for any fixed  $m$ ) is of Ackermannian growth. This completes the proof.  $\square$

**Remark 4.2.** Since the argument above only requires  $f$  to be defined on

$$(\{m\} \cup B_m^{n,m})^{[2]},$$

it follows (by “translation”) that  $g(n, m) \leq m + |B_m^{n,m}|$ .

That  $G(4, m) = 2^m(m + 2) - 1$  is shown in Fact 5.3 and the upper bound on  $g(5, \cdot)$  is shown in Theorem 7.1. Using this (all I need is that  $G(4, m)$  has exponential rate of growth) and the argument of Theorem 4.1, Theorem 1.2.3 follows easily:

**Proof of Theorem 1.2.3.** Use the notation of the proof above, and argue by induction on  $n \geq 5$  since the result is clear for  $n \leq 4$  from the explicit formulas for  $G(n, \cdot)$ . Notice the easy estimate  $l! < 2^{l(l-1)/2}$  and the obvious inequality  $s(i + 1, n, m) = s_{i+1} \leq G(n - 1, s_i!)$  for  $i < m$ . From this and Fact 5.3 we have that for  $n = 5$  there is a constant  $c_5$  such that  $s_i$  is bounded by a tower of two's of length  $c_5 i$  applied at  $m$ ,

$$s_i \leq 2^{2^{\cdot^{\cdot^{2^m}}}}.$$

In fact any  $c_5$  slightly larger than 3 suffices (with room to spare). This proves the result for  $n = 5$ ; for  $n > 5$  use Lemma 2.5 and proceed by a straightforward induction to show that  $c_{n-1} = n - 1$  suffices (and therefore for each  $m$ ,  $g(\cdot, m)$  has rate of growth precisely Ackermannian).  $\square$

**Question 4.3.** *Can the value of the constants  $c_n$  be significantly improved? This seems to require a more careful analysis than the one above, perhaps combined with fine detail considerations as in the proof of Theorem 7.1.*

## 5 Lower bounds for $g$ and $G$

Here I prove Theorem 1.3.

**Theorem 5.1.** 1.  $G(n + 1, m) \geq g^m(n, m + 1)$ .

2.  $g(n + 1, m + 1) \geq g(n, g(n + 1, m) + 1)$ . In particular, for  $n \geq 2$  and  $m \geq 1$ ,  $g(n, m) \geq A_{n-1}(m - 1)$ , the inequality being strict for  $n > 2$  and, for example,  $g(4, m) > 2^{m+2}$  for  $m > 1$ .

*Proof.* I exhibit a regressive function  $f : [m, g^m(n, m + 1) - 1]^{[2]} \rightarrow \mathbb{N}$  without min-homogeneous sets of size  $n + 1$  whose minimum element is  $m$ . Start by choosing regressive functions

$$F_k : [g^k(n, m + 1), g^{k+1}(n, m + 1) - 1]^{[2]} \rightarrow \mathbb{N}$$

without min-homogeneous sets of size  $n$ , for  $k < m$ ; this is possible by definition of  $g(n, \cdot)$ . Now set, for  $m < a \leq g^m(n, m + 1) - 1$ ,

$$f(m, a) = k \iff g^k(n, m + 1) \leq a < g^{k+1}(n, m + 1),$$

and for such  $a$  and  $b \in (a, g^{k+1}(n, m + 1) - 1]$ ,

$$f(a, b) = F_k(a, b).$$

Define  $f(a, b)$  for other values of  $a$  and  $b$  arbitrarily (below  $a$ ). This function works, for if  $\min(H) > m$  and  $\{m\} \cup H$  is min-homogeneous for  $f$ , then  $H$  is

completely contained in some interval  $[g^k(n, m + 1), g^{k+1}(n, m + 1))$  for some  $k < m$ , but then  $H$  is min-homogeneous for  $F_k$ , so  $|H| < n$ .

I now prove item 2. Let  $F_m : [m, g(n + 1, m)]^{[2]} \rightarrow \mathbb{N}$  be a regressive function without min-homogeneous sets of size  $n + 1$  and let

$$h_m : [g(n + 1, m) + 1, g(n, g(n + 1, m) + 1)]^{[2]} \rightarrow \mathbb{N}$$

be a regressive function without min-homogeneous sets of size  $n$ . Define  $F_{m+1} : [m + 1, g(n, g(n + 1, m) + 1)]^{[2]} \rightarrow \mathbb{N}$  by

$$F_{m+1}(a, b) = \begin{cases} F_m(a - 1, b - 1) & \text{if } b \leq g(n + 1, m), \\ a - 1 & \text{if } a \leq g(n + 1, m) < b, \\ h_m(a, b) & \text{if } g(n + 1, m) < a. \end{cases}$$

Then  $F_{m+1}$  is regressive. If  $H$  is min-homogeneous for  $F_{m+1}$  and  $|H| \geq 2$ , let  $a = \min(H)$  and  $b = \min(H \setminus \{a\})$ . If  $b \leq g(n + 1, m)$  then  $F_{m+1}(a, b) = F_m(a - 1, b - 1) < a - 1$  so  $H \subseteq [m + 1, g(n + 1, m)]$  and  $\{h - 1 : h \in H\}$  is min-homogeneous for  $F_m$ , so  $|H| \leq n$ .

If  $g(n + 1, m) < b$  then  $H \setminus \{a\}$  is min-homogeneous for  $h_m$ , so  $|H \setminus \{a\}| < n$  and  $|H| < n + 1$  in this case as well.  $\square$

**Remark 5.2.** Notice that for  $n = 3$ , the argument of Theorem 5.1.1 describes (up to trivial renamings) all the examples of regressive functions  $f : [m, g^m(3, m + 1) - 1]^{[2]} \rightarrow \mathbb{N}$  not admitting min-homogeneous sets of size 4 with minimum element  $m$ . It is easy now to give an example of a regressive  $f : [2, 14]^{[2]} \rightarrow \mathbb{N}$  witnessing  $14 \not\prec (5)_{reg}^2$ :

$$f(i, j) = \begin{cases} j - i - 1 \pmod{i} & \text{if } i \geq 6, \\ 0 & \text{if } \begin{array}{l} i = 2 \text{ and } j \leq 6, \\ i \in [3, 5] \text{ and } j = i + 1, \end{array} \\ 1 & \text{if } \begin{array}{l} i = 2 \text{ and } 7 \leq j, \\ i = 3 \text{ and } j \in \{5, 7, 8\}, \\ i \in \{4, 5\} \text{ and } j = i + 1, \end{array} \\ 2 & \text{if } \begin{array}{l} i = 3 \text{ and } j \in \{6\} \cup [9, 14], \\ i = 4 \text{ and } j = 7, \\ i = 5 \text{ and } 8 \leq j, \end{array} \\ 3 & \text{if } i = 4 \text{ and } 8 \leq j. \end{cases}$$

I leave to the reader the easy verification that this example works; in Theorem 6.1.2, I analyze a more difficult example witnessing  $g(4, 3) \geq 37$ . See Blanchard [1] for an analysis of a different example also witnessing  $g(4, 2) \geq 15$ , the function I have presented is closer in spirit to the other constructions in this paper.

Now I prove Theorem 1.2.1:

**Fact 5.3.**  $G(4, m) = 2^m(m + 2) - 1$ .

*Proof.* Notice that  $2^m(m+2) - 1 = g^m(3, m+1) \leq G(4, m)$  by Theorem 5.1.1. Suppose  $f : [m, 2^m(m+2) - 1]^{[2]} \rightarrow \mathbb{N}$  is regressive. A straightforward induction on  $k \leq m$  shows that either  $f(m, \cdot) \upharpoonright [m+1, 2^k(m+1) + 2^k - 1]$  takes at least  $k+1$  values or else  $f$  admits a min-homogeneous set  $A \in [m, 2^k(m+1) + 2^k - 1]^{[4]}$  with  $m \in A$  (see also the proof of Theorem 6.1.1 for a more detailed presentation of a similar approach). When  $k = m$ , this shows that  $G(4, m) \leq 2^m(m+2) - 1$ .  $\square$

**Remark 5.4.** Thus,  $g(4, 2) = G(4, 2) = 15$ . In the next section I improve the upper bound for  $g(4, m)$ ,  $m > 2$ .

**Corollary 5.5.**  $g(5, 2) > 2^{18}$ .

This significantly improves the bound  $g(5, 2) \geq 195$  claimed in Blanchard [1].

*Proof.*  $g(5, 2) \geq g(4, g(5, 1) + 1) = g(4, 16) > 2^{18}$ .  $\square$

**Remark 5.6.** In fact, by Theorem 6.1.2,  $g(4, 3) = 37$ , so  $g(4, m) \geq 5 \times 2^m - 3$  for  $m \geq 3$  and  $g(5, 2) \geq 5 \times 2^{16} - 3$ .

Theorem 5.1.2 also improves significantly the bound  $g(81, 2) > f_{51}(2^{2^{2^4}})$  obtained in Kojman *et al.* [6, Claim 2.32] (here,  $f_{51}$  is as in Section 2; to see that the new bound is an improvement, a slightly more precise version of Lemma 2.5 is necessary).

## 6 Bounds for $g(4, \cdot)$

From Section 5 it follows that  $g(4, m) \leq 2^m(m+2) - 1$ . Here I improve this bound and prove Theorem 1.4.

**Theorem 6.1.** 1. For  $m \geq 2$ ,  $g(4, m) \leq 2^m(m+2) - 2^{m-1} + 1$ .

2.  $g(4, 3) = 37$ .

3.  $g(4, 4) \leq 85$ .

*Proof.* I have already shown that  $g(4, 2) = 15$ . Assume  $m \geq 3$ , let  $n = 2^m(m+2) - 2^{m-1} + 1$  and suppose a regressive  $f : [m, n]^{[2]} \rightarrow \mathbb{N}$  is given. I need to argue that there is  $H \in [m, n]^{[4]}$  min-homogeneous for  $f$ . For  $i < m$  let  $a_i = \min\{j : f(m, j) = i\}$  and  $C_i = \{j > a_i : f(m, j) = i\}$ . One may assume that, as long as the  $a_i$  are defined, they occur in order, so  $m+1 = a_0 < a_1 < \dots$

If  $f(m+1, a) = f(m+1, b)$  for  $a \neq b$  in  $C_0$  then  $H = \{m, m+1, a, b\}$  is as required. Assume now that  $f(m+1, \cdot) \upharpoonright C_0$  is injective and, in particular,  $|C_0| \leq m+1$ .

For  $i \in C_0$  let  $B_i = \{j > i : f(m+1, j) = f(m+1, i)\}$ . I claim that for all  $k \in [1, m-2]$ , either  $a_k \leq 2^k(m+2) - 2^{k-1} - 1$  or else there is an  $H$  as required and either of the form  $\{m, a_i, a, b\}$  for some  $i < k$  and some  $a, b \in C_i$  or of the form  $\{m+1, i, a, b\}$  for some  $i \in C_0$  and some  $a, b \in B_i$ .

The proof is by induction on  $k$ . Fix a least counterexample. Then  $a_t \leq 2^t(m+2) - 2^{t-1} - 1$  for all  $t \in [1, k)$  and  $1 \leq k < m-1$ . Then  $a_k \leq 2^k(m+2) - 2^{k-1}$ . Otherwise, for some  $i < k$ ,  $|C_i| > a_i$ . If  $a_k = 2^k(m+2) - 2^{k-1}$ , then  $a_t = 2^t(m+2) - 2^{t-1} - 1$  for all  $t \in [1, k)$  (or else, again, some  $C_i$  for  $i < k$  has size larger than  $a_i$ ). Also, there is some  $j \in (2m+1, a_k)$  in  $C_0$ . But then  $|B_i| > i$  for some  $i \in C_0$  and the claim follows: Otherwise,

$$\begin{aligned} \sum_{i \in C_0} |B_i| &\leq \sum_{i \in [m+2, 2m+1] \cup \{j\}} i \leq \sum_{i=m+2}^{2m+1} i + 2^k(m+2) - 2^{k-1} - 1 \\ &= \frac{3}{2}m(m+1) + 2^k(m+2) - 2^{k-1} - 1 \\ &< n - 2(m+1) = |[2m+2, n] \setminus \{j\}| \end{aligned}$$

because  $(3+2m)(2^m - 2^k) \geq 3(3+2m)2^{m-2} > 3m^2 + 7m$  for  $m \geq 3$ .

It follows that one may assume  $a_{m-1} \leq 2^{m-1}(m+2) - 2^{m-2}$ , but then, since  $n \geq 2a_{m-1} + 1$ , some  $C_i$  must have size larger than  $a_i$ , and the proof is complete.

Now I show that  $g(4, 3) = 37$ . The upper bound follows from the argument above. To see that  $g(4, 3) \geq 37$  I exhibit a regressive  $f : [3, 36]^{[2]} \rightarrow \mathbb{N}$  without min-homogeneous sets of size 4. Consider the function  $f$  shown below: For

$3 \leq i < j \leq 36$ , set

$$f(i, j) = \left\{ \begin{array}{ll} j - i - 1 \pmod{i} & \text{if } \begin{array}{l} i \geq 16, \\ 8 \leq i \leq 15 \text{ and } j \leq 16, \\ 12 \leq i \leq 15 \text{ and } j \leq 19, \\ 4 \leq i \leq 6 \text{ and } j \leq 7, \\ i = 6 \text{ and } j \leq 11, \end{array} \\ 0 & \text{if } \begin{array}{l} i = 3 \text{ and } (j \leq 7 \text{ or } j = 17), \\ i = 5 \text{ and } 8 \leq j \leq 11, \\ i = 6 \text{ and } 12 \leq j \leq 16, \\ i = 7 \text{ and } j \leq 12, \end{array} \\ 1 & \text{if } \begin{array}{l} i = 3 \text{ and } 8 \leq j \leq 16, \\ i = 4 \text{ and } 8 \leq j \leq 11, \\ i = 5 \text{ and } 12 \leq j \leq 16, \\ i = 6 \text{ and } j = 18, \\ i = 7 \text{ and } j = 13, \end{array} \\ 2 & \text{if } \begin{array}{l} i = 3 \text{ and } 18 \leq j, \\ i = 4 \text{ and } j \in [12, 19] \setminus \{17\}, \\ i = 5 \text{ and } j = 17, \\ i = 6 \text{ and } j = 19, \\ i = 7 \text{ and } j = 14, \\ i = 15 \text{ and } 21 \leq j, \end{array} \\ 3 & \text{if } \begin{array}{l} i = 4 \text{ and } (j = 17 \text{ or } 20 \leq j), \\ i = 5 \text{ and } 18 \leq j, \\ i = 7 \text{ and } j = 15, \\ i = 11 \text{ and } 17 \leq j \leq 20, \\ i = 14 \text{ and } 20 \leq j, \end{array} \\ 4 & \text{if } \begin{array}{l} i = 7 \text{ and } j = 16, \\ i = 10 \text{ and } 17 \leq j \leq 20, \\ i = 11 \text{ and } 21 \leq j, \\ i = 13 \text{ and } 20 \leq j, \\ i = 15 \text{ and } j = 20, \end{array} \\ 5 & \text{if } \begin{array}{l} i = 6 \text{ and } (j = 17 \text{ or } 20 \leq j), \\ i = 7 \text{ and } (j = 17 \text{ or } j = 19), \\ i = 9 \text{ and } 17 \leq j \leq 20, \\ i = 10 \text{ and } 21 \leq j, \\ i = 12 \text{ and } 20 \leq j, \end{array} \\ 6 & \text{if } \begin{array}{l} i = 7 \text{ and } (j = 18 \text{ or } 20 \leq j), \\ i = 8 \text{ and } 17 \leq j \leq 20, \\ i = 9 \text{ and } 21 \leq j, \end{array} \\ 7 & \text{if } i = 8 \text{ and } 21 \leq j. \end{array}$$

To help understand the example somewhat, notice that the argument above

shows that one must have  $a_1 = 8$  and  $a_2 = 18$ ,  $f(i, \cdot)$  must be injective for  $i \geq 18$  and similarly  $f(i, \cdot) \upharpoonright C_i$  must be injective for  $i \in [4, 7]$  and  $C_i = \{j > i : f(3, j) = f(3, 4)\}$ , or  $i \in [8, 16] \cap \{j : f(3, j) = f(3, 8)\}$  and  $C_i = [i+1, 17] \cap \{j : f(3, j) = f(3, 8)\}$ . If  $f$  is any function satisfying these conditions,  $a < b < c < d$ , and  $A = \{a, b, c, d\}$  is min-homogeneous for  $f$ , then  $a > 3$  and  $b < 18$ .

The function  $f$  displayed above satisfies the conditions just described. Let  $A$  as above be a putative min-homogeneous set. Then  $a < 16$  since otherwise  $f(a, \cdot)$  does not take any value more than twice.

In fact,  $a < 12$ , since  $12 \leq a \leq 15$  would imply (for the same reason) that  $b \geq 18$ . If  $8 \leq a \leq 11$ , then  $b \geq 15$ . Since  $f(i, \cdot) \upharpoonright D_i$  is injective for  $i \in \{15\} \cup [17, 20]$  and  $D_i = (i, 20]$ , or  $i = 16$  and  $D_i = [21, 36]$ , this is not possible.

If  $a = 7$  then  $b \notin [8, 12]$  as  $f(i, \cdot) \upharpoonright (i, 12]$  is injective for  $i \in [8, 12]$ . This forces  $b \geq 18$ .

If  $a = 6$  then  $b \notin \{7\} \cup [12, 16]$  as  $f(b, \cdot) \upharpoonright [\max(b+1, 12), 16]$  is then injective. This forces  $b = 17$  but  $f(17, \cdot) \upharpoonright [20, 36]$  is injective, so this cannot be the case.

The analysis above already rules out  $a = 5$  since  $f(6, \cdot) \upharpoonright [8, 11]$  is injective. Since  $f(7, \cdot) \upharpoonright [12, 16] \cup \{18, 19\}$  is also injective, it also rules out  $a = 4$ , completing the argument.

Finally, I argue that  $g(4, 4) \leq 85$ . Let a regressive  $f : [4, 85]^{[2]} \rightarrow \mathbb{N}$  be given. Use notation as before. Then one can assume (from the argument for item 1) that  $a_1 \leq 10$ . If  $a_1 = 10$ , since  $6 + 7 + 8 + 9 = 30$ , one can assume that there is  $b \leq 40$  such that  $f(5, b) = 4$  (while  $f(5, j) = j - 6$  for  $j \in [6, 9]$ ). But then there is a min-homogeneous set for  $f$  of size 4 with minimum element 5 and maximum at most 81.

If  $a_1 \leq 9$  then  $a_2 \leq 21$ . If  $a_2 = 21$  then one can assume  $f(5, j) = j - 6$  for  $j \in [6, 8]$  and there are  $b_1, b_2$  with  $f(5, b_1) = 3$ ,  $f(5, b_2) = 4$ ,  $b_1 \leq 19$  and  $b_2 \leq 20$ . Since  $6 + 7 + 8 + 19 + 20 = 60$ , there is again a min-homogeneous set of size 4 in this case. If  $a_2 \leq 20$ , then  $a_3 \leq 42$  and  $|A_i| > a_i$  for some  $i < 4$ . This shows  $g(4, 4) \leq 85$ .  $\square$

## 7 Bounds for $g(5, \cdot)$

In this section I sketch briefly how to adapt the proof of Blanchard [1, Lemma 3.1] to prove the more general statement below, which concludes the proof of Theorem 1.2. The bound for  $g(5, 2)$  is smaller than the one in [1] because I take advantage of the fact that  $g(4, 3) = 37$  as established in Theorem 6.1.2.

**Theorem 7.1.** *Let  $m$  be given. For  $i < m$  set  $d_i = g^i(4, m + 1)$ . Let  $\alpha_{-1} = 0$  and  $\alpha_i = (\alpha_{i-1} + m + 3 + i)(2^{d_i} - 1)$  for  $0 \leq i < m$ . Then*

$$g(5, m) \leq (2m + 1) + \sum_{i=0}^{m-1} \alpha_i.$$

*In particular,  $g(5, 2) \leq 41 \times 2^{37} - 1$ .*

*Proof.* Let  $n$  be the purported upper bound displayed above and consider a regressive function  $f : [m, n]^{[2]} \rightarrow \mathbb{N}$ . For  $i < m$  let

$$B_i = \{x \in [m+1, n] : f(m, x) = i\}$$

and if  $B_i \neq \emptyset$ , set  $a_i = \min(B_i)$ . Without loss,  $a_0 = m+1 < a_1 < \dots$ . Clearly, we may assume that  $a_i \leq g^i(4, m+1) = d_i$  for all those  $i < m$  for which  $a_i$  is defined. In particular, since  $n$  is sufficiently large, we may assume that the  $a_i$  are defined for all  $i < m$ .

Consider  $B_{ij} = \{x \in [a_i+1, n] : f(m, x) = i, f(a_i, x) = j\}$  for  $i < m$  and  $j < a_i$ , and if  $B_{ij} \neq \emptyset$ , set  $a_{ij} = \min(B_{ij})$ . Let  $D = \{B_{ij} : B_{ij} \neq \emptyset\}$  and  $q = |D|$ , so  $q \leq \sum_{i=0}^{m-1} d_i$ . Let  $\{C_s : s < q\}$  be the enumeration of  $D$  such that, setting  $c_s = \min(C_s)$ , then the sequence  $(c_s : s < q)$  is strictly increasing.

Notice that  $a_i \notin C_l$  for any  $i, l$ , and  $a_i < a_{ij}$  for all  $i, j$  such that  $a_{ij}$  is defined. For  $i < m$  define  $k_i$  as the least  $k < q$  such that  $a_i < c_k$ . Then

$$k_i \leq \sum_{j=0}^{i-1} a_j \leq \sum_{j=0}^{i-1} d_j.$$

I now proceed to find an upper bound  $l_s$  on the size of  $C_s$  beyond which one is guaranteed to find a min-homogeneous set of size 5. The value of  $n$  displayed above is obtained by observing that

$$[m, n] = \{m\} \cup \{a_i : i < m\} \cup \bigcup_{s=0}^{q-1} C_s,$$

so  $n - m + 1 = m + 1 + \sum_{s=0}^{q-1} |C_s|$ , and setting  $n \geq 2m + \sum_s l_j + 1$ .

To find  $l_s$ , notice that

$$[m, c_s] \subseteq \{m\} \cup \{a_i : a_i < c_s\} \cup \bigcup_0^{s-1} C_j \cup \{c_s\},$$

so  $c_s - m + 1 \leq 2 + (i+1) + \sum_0^{s-1} |C_j|$ , where  $s \in [k_{i-1}, k_i)$ , or

$$c_s \leq m + 1 + (i+1) + \sum_0^{s-1} |C_j|.$$

Let  $C'_s = C_s \setminus \{c_s\}$ . If

$$|C'_s| \geq (m+2) + (i+1) + \sum_0^{s-1} |C_j|,$$

then  $f(c_s, \cdot) \upharpoonright C'_s$  is not injective, so there are  $d < e$  in  $C'_s$  such that  $f(c_s, d) = f(c_s, e)$  and  $\{m, a_j, c_s, d, e\}$  is min-homogeneous, where  $j \leq i$  is chosen so that  $C_s = B_{jk}$  for some  $k$ .

This gives the upper bound  $l_s \leq (m+i+3) + \sum_0^{s-1} l_j$  so, by a straightforward induction,

- $l_s \leq 2^s(m+3)$  for  $s < d_0$ ,
- $l_s \leq 2^{s-d_0}((m+3)(2^{d_0}-1) + (m+4))$  for  $d_0 \leq s < d_0 + d_1$ ,
- and, in general, for  $i < m$ ,  $l_s \leq 2^{s-d_{i-1}}(\dots((m+3)(2^{d_0}-1) + (m+4))(2^{d_1}-1) + \dots)(2^{d_{i-1}}-1) + (m+3+i)$  for  $\sum_{j=0}^{i-1} d_j \leq s < \sum_{j=0}^i d_j$ .

These upper bounds give the value of  $n$  that I started with, and the claimed inequality  $g(5, m) \leq n$  follows. In the case  $m = 2$ , it implies

$$\begin{aligned} g(5, 2) &\leq (2 \times 2 + 1) + (2 + 3)(2^{2+1} - 1) + (5(2^3 - 1) + 6)(2^{g(4,3)} - 1) \\ &= 40 + 41(2^{37} - 1) = 41 \times 2^{37} - 1. \end{aligned}$$

This completes the proof.  $\square$

I conclude with some questions:

**Question 7.2.** *Is  $G(n+1, m) > g^m(n, m+1)$  for  $n > 4$ ?*

**Question 7.3.** *Is  $2^m(m+1) \leq g(4, m)$  for all  $m$ ?*

The proof of Theorems 6.1 and 7.1 suggest that to fully understand  $g$  requires to solve the following question:

**Question 7.4.** *What is the rate of growth of the following function  $k(n, m)$ ? For any  $n, m$  and regressive  $f : [m, g(n, m)]^{[2]} \rightarrow \mathbb{N}$  set*

$$k_f = \min\{\min(H) : H \in [m, g(n, m)]^{[n]} \text{ is min-homogeneous for } f\}$$

and let

$$k(n, m) = \max\{k_f : f : [m, g(n, m)]^{[2]} \rightarrow \mathbb{N} \text{ is regressive}\}.$$

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