Itay Neeman. ***The Determinacy of Long Games.*** De Gruyter Series in Logic and its Applications, vol. 7. Walter de Gruyter, Berlin, 2004 xi + 317 pp.

The deep connection between large cardinals and determinacy has been a central concern of modern research in set theory. The simplest possible setting is easy to recall: A set $C \subset \omega^\omega$ is fixed, and two players I and II alternate choosing integers $x(n)$ for infinitely many stages, with player I playing first, thus defining a real $x = \langle x(n) \colon n \in \omega\rangle \in \omega^\omega$. We can call these infinitely many moves a *round*. Player I wins this round iff $x \in C$, otherwise player II wins. Let's denote this game by $G_\omega(C)$. A winning strategy for I is a function $\tau : \omega^{<\omega} \to \omega$ such that if at each stage n player I plays $x(2n) = \tau(\langle x(0), \dots, x(2n-1)\rangle)$, then the resulting real x is in C. Similarly we can define a winning strategy for player II, and $G_\omega(C)$ is *determined* iff one of the players has a winning strategy.

From a well-ordering of the reals it is easy (by a diagonal argument) to produce a non-determined set of reals. However, large cardinal axioms imply that all sets of reals in $L(\mathbb{R})$, and more, are determined. See, for example, Neeman's papers *Optimal proofs of determinacy*, ***The bulletin of symbolic logic***, vol. 1 (1995), pp. 327–339 and *Optimal proofs of determinacy II*, ***The journal of mathematical logic***, vol. 2 (2002), pp. 227–258 (from now on, OpdII). Conversely, from the determinacy of sufficiently closed pointclasses of reals, the existence of inner models with large cardinals can be established. See, for example, the forthcoming paper by Koellner and Woodin, *Large cardinals from determinacy*, to appear in the ***Handbook of Set Theory***, Kanamori, Foreman, Magidor, eds. All of this constitutes what one could call the *first generation* of results tying large cardinals and determinacy. This is of course informal notation, and not intended to convey the impression that nothing else can be said or done about first generation results.

The second generation of results, to which the book under review is a seminal contribution, probably begins with John Steel's paper *Long games*, in ***Cabal Seminar 81–85***, Kechris, Martin, Steel, eds., Springer (1988), pp. 56–97 (from now on, Lg). In the presence of large cardinal axioms, the first generation results focus on obtaining the determinacy of more and more complex real pointclasses, where the complexity is measured descriptive set theoretically. Here, the attention shifts from the complexity of the classes to that of the games themselves, by allowing them to run for longer than ω many stages.

This book is intended for a serious graduate student or a researcher in the area. It is very carefully written and although a serious effort has been made to motivate its arguments, they tend to be very long and technical, and a novice may feel lost at points. In *An introduction to proofs of determinacy of long games*, in ***Logic Colloquium '01, Lecture Notes in Logic, vol. 20***, Baaz, Friedman, Krajícek, eds., AK Peters (2005), pp. 43–88, Neeman has written a wonderful expository account of the theory described in this book, leaving out the technical details that comprise most of the book itself, and the reader is encouraged to look at this paper first, as a motivation for the work to come.

In terms of prerequisites for these technical details, the two papers by Martin and Steel JSL LVII 1332 and JSL LVII 1333, and Neeman's *Inner models in the region of a Woodin limit of Woodin cardinals*, ***Annals of pure and applied logic***, vol. 116 (2002), pp. 67–155 (from now on, ImrWlWc) are essential, although if the reader is familiar with the basic theory of iteration trees and is willing to accept some results as black boxes, then Appendix A of the book will suffice.

The book begins with a lively historical introduction in which basic definitions are recalled. It explains how the central large cardinal concept of *Woodin cardinal* was isolated and why it is the key notion in determinacy results. In their pivotal papers cited above, Martin and Steel introduced the notion of *iteration tree*, which is also at the core of modern inner model theory. In short, nice inner models for large cardinal notions weaker than Woodin cardinals can be compared by iterating ultrapowers, very much as in Kunen's argument for models with measurable cardinals, see e.g. Jech, BSL XI 243. These are essentially *linear* iterations, and this linearity seriously bounds the complexity of the reals that can belong to such inner models (for example, in the fine structural context, all these reals are Δ^1_3 in a countable ordinal). Martin and Steel found a non-linear method of iterating ultrapowers of inner models with Woodin cardinals. These models give then rise to iteration trees, trees of structures with embeddings between the models appearing along their branches. If two models are compared this way, at limit stages of the comparison process, different possibilities on how to continue the trees may arise, and the existence of these choices increases the complexity of their comparison process and explains why these models allow more complicated reals than those appearing in linearly iterable models.

The comparison process between models with Woodin cardinals may be naturally described in terms of *iteration games* between two players, "good" and "bad". There are several kinds of iteration games, but essentially all consist of "bad" playing iteration trees and then "good" picking a branch leading to a well-founded model, which "bad" then uses as the root of a new tree to continue the game, with this going on for as many rounds as the game specifies. The game ends if an ill-founded model is produced, otherwise "good" wins. If "good" has a winning strategy, then the original model at the root of the first tree played by "bad" is *iterable* and the strategy is called an *iteration strategy.* Neeman's ImrWIWc shows that the existence of enough large cardinals implies the existence of *nice* models M, i.e., iterable models of enough set theory with as many Woodin cardinals as required for any of the applications in the book.

The key idea is now that these iteration strategies can be used to produce winning strategies for either I or II in *long* games of the kind one considers when interested in determinacy by means of what one could call a *translation procedure.* The simplest of these long games, $G_{\omega\cdot(n+1)}(C)$, for $C \subset (\omega^\omega)^{n+1}$, is defined similarly to $G_\omega(C)$, only that now $n+1$ rounds are played, one after the other, thus producing an $(n+1)$-tuple of reals $\langle x_0, \ldots, x_n \rangle$. As before, player I wins iff this tuple is in C. Say that C is Π^1_1. Then the determinacy of $G_{\omega\cdot(n+1)}(C)$ implies the determinacy of all the games $G_\omega(D)$ for D in Π^1_{n+1}, since the quantifiers in the projective definition of D can be simulated with the runs that give rise to $x_1, \ldots, x_n$.

The core of the book is a series of arguments showing how to translate iteration strategies for nice models M as above into winning strategies for either player in one of several possible kinds of long games. For the games $G_{\omega\cdot(n+1)}(C)$ just explained, this is described in detail in chapter 1, which also presents a concise introduction to the theory of Woodin cardinals. This argument strongly resembles the one in OpdII, and the reader may want to look at this paper as a warm-up. This chapter also introduces, in a simple setting, the notation that is required to understand subsequent chapters. I must say that at some points the author's choice of notation seemed to me slightly odd, but no non-standard usage occurs, and the presentation is detailed and careful enough that the reader may just as well ignore my comment.

Several games are introduced, they are each more elaborate than the ones from previous chapters. The games described in chapter 1 are thus designed to provide a proof of projective determinacy by showing the determinacy of games $G_{\omega\cdot(n+1)}(C)$ for C closed. In chapter 2 determinacy is shown for games of fixed countable length, and in chapter 3 this is generalized even further to games of continuously varied countable length. These correspond, and generalize, the long games of Steel's paper Lg. Finally, chapters 4 through 7 are devoted to the proof of determinacy of games whose length reaches a local cardinal, i.e., this length is uncountable in some inner model, but not necessarily in V. As a typical application, Exercise 7.F.14 shows the result of Woodin that (from the existence of an iterable M such that there is a countable θ that in M is Woodin and limit of Woodin cardinals), there is a proper class inner model P such that, in P, all definable games of length ω_1^P are determined. The techniques of the book have been further generalized in subsequent papers by Neeman, so this area of research is clearly very promising territory.

Along the way, the reader encounters a few additional topics of independent interest. For example, universally Baire sets and homogeneously Suslin sets are discussed in chapter 2, and in chapter 4 a very general presentation of Woodin's extender algebra (in many generators) is given. The only presentation of the extender algebra that I was aware of was what would be in this context the algebra in one generator. See, for example, Steel *An outline of inner model theory*, to appear in the ***Handbook of Set Theory***, Kanamori, Foreman, Magidor, eds.

Let me finally say a few words about the translation procedure. Throughout the book the setting consists of a nice inner model M as above, a set $\mathcal{W}$ of Woodin cardinals from M (there are some restrictions on the Woodin cardinals in $\mathcal{W}$, but we can ignore them here), some $\delta \in \mathcal{W}$ and some $\mathtt{col}(\omega, \delta)$-name $\dot{A}$ coding what is in essence a set of reals (associated to the set we have been calling C). Actually, $\dot{A}$ is a name for a subset of $(M\|\delta)^\omega \times \omega^\omega$, where $M\|\delta$ denotes V_δ^M. Essential to the arguments is the continuity of a series of approximating games $\mathcal{A}[s]$ for $s \in \omega^{<\omega}$. Continuity is important in that these games are given inside of M although they give rise to games $\mathcal{A}[x]$ for $x \in \omega^\omega$. This matters for two reasons. An obvious one is that sometimes $x \notin M$. A more subtle one is that M is replaced frequently by other models N coming from iteration trees on M, so the games $\mathcal{A}[s]$, being elements of M, can be moved into the corresponding new games on N via the same embeddings that move M. The games $\mathcal{A}[s]$ build on the argument from Martin and Steel, JSL LVII 332 and, as in Neeman's game in OpdII, either a winning strategy exists for player I in M for $\mathcal{A}[x]$ or a winning strategy exists for player II in an ultrapower of M for a game corresponding to the images of δ and of $\dot{A}$ under this ultrapower. The game $\mathcal{A}[x]$ consists of player I trying to witness that $x \in \dot{A}[h]$ for h generic over M with II trying to witness the opposite. Greatly simplifying what is actually done, in each stage, player I plays conditions in $\mathtt{col}(\omega, \delta)$ and a set of names for reals. These collections of names are decreasing, the conditions force that any name in the collection should be in $\dot{A}$, and that these names agree more and more with x. Player II plays dense sets and the collections that player I plays should be contained in these dense sets, thus eventually producing a generic for $\mathtt{col}(\omega, \delta)$ as required if player I has a winning strategy.

The key notions that the book introduces are those of *pivot* and *mixing* games. The pivot games, intended in the case where player I has no winning strategy, build in addition an iteration tree on M (using extenders below δ), and a continuous witness that the relevant branch on which the outcome for player II is examined is well-founded. This is done using a key lemma from Martin and Steel, JSL LVII 332. Mixing games give player I some flexibility on how the iteration trees must be extended and in particular restrict the possible extenders to be used along the way. The pivot game is designed so that either player I has a winning strategy in $\mathcal{A}[x]$ or else player II ought to have a winning strategy in the *shifted* game. This is by no means automatic. For example, chapter 5 is devoted to what amounts to a proof that this is the case for the long games to be presented in chapter 7. What can ultimately be described as a correctness argument for M shows that this implies that either I or II has a winning strategy in the original game on C. The way these pivot and mixing games are combined is quite delicate. The extender algebra is used in some cases to produce reals that guide how these games are executed. The iterability of M is essential to even describe how the games on M corresponding to continuously played games are to be organized.

To close, let me add that this book is a very welcome addition to the literature in set theory and a magnificent example of the high standards that the de Gruyter Series is intending to achieve. It is by no means an easy book, but (as I mentioned above) it is very carefully written, its arguments are in essence self-contained, and motivation is presented, even though somewhat more sparingly as the book progresses; both the techniques and even the definitions are quite involved, but a patient and careful reader will surely be more than handsomely rewarded.

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